BOHR/LEVITAN ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITHOUT FAVARD'S SEPARATION CONDITION.

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ABSTRACT. We prove that the linear stochastic equation $dx(t) = (A(t)x(t) + f(t))dt + g(t)dW(t)$ with linear operator $A(t)$ generating a continuous linear cocycle φ and Bohr/Levitan almost periodic or almost automorphic coefficients $(A(t), f(t), g(t))$ admits a unique Bohr/Levitan almost periodic (respectively, almost automorphic) solution in distribution sense if it has at least one precompact solution on $\mathbb{R}_+$ and the linear cocycle φ is asymptotically stable.

Dedicated to the memory of Professor **V. V. Zhikov**.

1. INTRODUCTION

This paper is dedicated to the study of linear stochastic differential equations with Bohr/Levitan almost periodic and almost automorphic coefficients. This field is called Favard's theory [20, 33], due to the fundamental contributions made by J. Favard [15]. In 1927, J. Favard published his celebrated paper, where he studied the problem of existence of almost periodic solutions of equation in $\mathbb{R}$ of the following form:

$$x' = A(t)x + f(t) \tag{1}$$

with the matrix $A(t)$ and vector-function $f(t)$ almost periodic in the sense of Bohr (see, for example, [16, 20]).

Along with equation (1), consider the homogeneous equation

$$x' = A(t)x$$

and the corresponding family of limiting equations

$$x' = B(t)x, \tag{2}$$

where $B \in H(A)$, and $H(A)$ denotes the hull of almost periodic matrix $A(t)$ which is composed by those functions $B(t)$ obtained as uniform limits on $\mathbb{R}$ of the type $B(t) := \lim_{n\to\infty} A(t + t_n)$, where $\{t_n\}$ is some sequence in $\mathbb{R}$.

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Theorem 1.1. *(Favard's theorem* [15]*) The linear differential equation (1) with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution, and each bounded solution $\varphi(t)$ of every limiting equation (2) ($B \in H(A)$) is separated from zero, i.e.*

$$\inf_{t\in\mathbb{R}} |\varphi(t)| > 0. \tag{3}$$

This result was generalized infinite-dimensional equation in the works of V. V. Zhikov and B. M. Levitan [33] (see also B. M. Levitan and V. V. Zhikov [20, ChVIII]).

Favard's theorem for linear differential equation with Levitan almost periodic (respectively, almost automorphic) coefficients was established by B. M. Levitan [19, ChIV] (respectively, by Lin F. [21]).

For linear stochastic differential equation Favard's theorem was established by Liu Z. and Wang W. in [22].

In the work [2] it was proved that Favard's theorem remains true if we replace condition (3) by the following:

$$\inf_{t\to+\infty} |\varphi(t)| = 0. \tag{4}$$

In this paper we establish that Favard's theorem remains true for linear stochastic differential equations under the condition (4).

This paper is organized as follows.

In Section 2 we collect some well known facts from the theory of dynamical systems (both autonomous and non-autonomous). Namely, the notions of almost periodic (both in the Bohr and Levitan sense), almost automorphic and recurrent motions; cocycle, skew-product dynamical system, and general non-autonomous dynamical system, comparability of motions by character of recurrence in the sense of Shcherbakov [26]-[28].

Section 3 is dedicated to the proof of classical Birkoff's theorem (about existence of compact minimal set) for non-autonomous dynamical systems.

In Section 4 we study the problem of strongly comparability of motions by character of recurrence of semi-group non-autonomous dynamical systems. The main result is contained in Theorem 4.11 and it generalizes the known Shcherbakov's result [26, 28].

Section 5 is dedicated to the shift dynamical systems and different classes of Poisson stable functions. In particular: quasi-periodic, Bohr almost periodic, almost automorphic functions and many others.

In Section 6 we collect some results and constructions related with Linear (homogeneous and nonhomogeneous) Differential Systems. We also discusses here the relation between two definitions of hyperbolicity (exponential dichotomy) for linear non-autonomous systems. The main result of this section (Theorem 6.13) establish the equivalence of two definitions for finite-dimensional and for some classes of infinite-dimensional systems.

Section 7 is dedicated to the study of Bohr/Levitan almost periodic and almost automorphic solutions of Linear Stochastic Differential Equations. The main results (Theorems 7.8, 7.13 and Corollaries 7.9, 7.14) show that classical Favard's theorem remains true (under some conditions) for linear stochastic differential equations.

2. Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems

Let X be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of nonnegative real (integer) numbers, $\mathbb{T}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{S} \subseteq \mathbb{T}$ ($\mathbb{T}_+ \subseteq \mathbb{S}$) be a sub-semigroup of the additive group $\mathbb{T}$, where $\mathbb{T}_+ := \{t \in \mathbb{T} : \ t \geq 0\}$.

Let $(X, \mathbb{S}, \pi)$ be a dynamical system.

Definition 2.1. *Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ ($\mathbb{T}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{T}$) be two dynamical systems. A mapping $h : X \to Y$ is called a homomorphism (isomorphism, respectively) of the dynamical system $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, if the mapping h is continuous (homeomorphic, respectively) and $h(\pi(x, t)) = \sigma(h(x), t)$ ($t \in \mathbb{T}_1, \ x \in X$). In this case the dynamical system $(X, \mathbb{T}_1, \pi)$ is an extension of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ by the homomorphism h, but the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called a factor of the dynamical system $(X, \mathbb{T}_1, \pi)$ by the homomorphism h. The dynamical system $(Y, \mathbb{T}_2, \sigma)$ is called also a base of the extension $(X, \mathbb{T}_1, \pi)$.*

Definition 2.2. *A triplet $\langle (X, \mathbb{T}_1, \pi), \ (Y, \mathbb{T}_2, \sigma), \ h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, is called a non-autonomous dynamical system (NDS).*

Definition 2.3. *A triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ (or shortly φ), where $(Y, \mathbb{T}_2, \sigma)$ is a dynamical system on Y, W is a complete metric space and φ is a continuous mapping from $\mathbb{T}_1 \times W \times Y$ to W, satisfying the following conditions:*

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1, \ u \in W, y \in Y)$,

is called [24] *a cocycle on $(Y, \mathbb{T}_2, \sigma)$ with the fiber W.*

Definition 2.4. *Let $X := W \times Y$ and define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as following: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e., $\pi = (\varphi, \sigma)$). Then it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X which is called a skew-product dynamical system* [24] *and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, consequently, $\langle (X, \mathbb{T}_1, \pi), \ (Y, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.*

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ $(X := W \times Y)$ called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of theses.

Example 2.5. Let E be a real or complex Banach space and Y be a metric space. Denote by $C(Y\times E,E)$ the space of all continuous mappings $f:Y\times E\mapsto E$ endowed by compact-open topology. Consider the system of differential equations

$$\begin{cases} u' = & F(y,u) \\ y' = & G(y), \end{cases} \tag{5}$$

where $Y\subseteq E, G\in C(Y,E)$ and $F\in C(Y\times E,E)$. Suppose that for the system (5) the conditions of the existence, uniqueness, continuous dependence of initial data and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $(Y,\mathbb{R}_+,\sigma)$ a dynamical system on Y generated by the second equation of the system (5) and by $\varphi(t,u,y)$ – the solution of equation

$$u' = F(yt,u)\ (yt := \sigma(t,y)) \tag{6}$$

passing through the point $u\in E$ for $t=0$. Then the mapping $\varphi:\mathbb{R}_+\times E\times Y\to E$ is continuous and satisfies the conditions: $\varphi(0,u,y)=u$ and $\varphi(t+\tau,u,y)=\varphi(t,\varphi(\tau,u,y),yt)$ for all $t,\tau\in\mathbb{R}_+$, $u\in E$ and $y\in Y$ and, consequently, the system (5) generates a non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$ (where $X:=E\times Y$, $\pi:=(\varphi,\sigma)$ and $h:=pr_2:X\to Y$).

We will give some generalization of the system (5). Namely, let $(Y,\mathbb{R}_+,\sigma)$ be a dynamical system on the metric space Y. Consider the system

$$\begin{cases} u' = & F(yt,u) \\ y\in & Y, \end{cases} \tag{7}$$

where $F\in C(Y\times E,E)$. Suppose that for the equation (6) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. The system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$, where $X:=E\times Y$, $\pi:=(\varphi,\sigma)$, $\varphi(\cdot,u,y)$ is the solution of (6) and $h:=pr_2:X\to Y$ is a non-autonomous dynamical system generated by the equation (7).

Example 2.6. Let us consider a differential equation

$$u' = f(t,u), \tag{8}$$

where $f\in C(\mathbb{R}\times E,E)$. Along with equation (8) we consider its H-class [1],[20], [24], [28], i.e., the family of equations

$$v' = g(t,v), \tag{9}$$

where $g\in H(f):=\overline{\{f^{\tau}:\tau\in\mathbb{R}\}}$, $f^{\tau}(t,u):=f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times E$ and by bar we denote the closure in $C(\mathbb{R}\times E,E)$. We will suppose also that the function f is regular, i.e. for every equation (9) the conditions of the existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled. Denote by $\varphi(\cdot,v,g)$ the solution of equation (9) passing through the point $v\in E$ at the initial moment $t=0$. Then there is a correctly defined mapping $\varphi:\mathbb{R}_+\times E\times H(f)\to E$ satisfying the following conditions (see, for example, [1], [24]):

1) $\varphi(0,v,g)=v$ for all $v\in E$ and $g\in H(f)$;
2) $\varphi(t,\varphi(\tau,v,g),g^{\tau})=\varphi(t+\tau,v,g)$ for every $v\in E$, $g\in H(f)$ and $t,\tau\in\mathbb{R}_+$;
3) the mapping $\varphi:\mathbb{R}_+\times E\times H(f)\to E$ is continuous.

Denote by $Y:=H(f)$ and $(Y,\mathbb{R}_+,\sigma)$ a dynamical system of translations (a semigroup system) on Y, induced by the dynamical system of translations $(C(\mathbb{R}\times$

$E,E),\mathbb{R},\sigma)$. The triplet $\langle E,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ is a cocycle on $(Y,\mathbb{R}_+,\sigma)$ with the fiber E. Thus, equation (8) generates a cocycle $\langle E,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ and a non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),\ (Y,\mathbb{R}_+,\sigma),h\rangle$, where $X := E\times Y$, $\pi := (\varphi,\sigma)$ and $h := pr_2 : X\to Y$.

Remark 2.7. *Let $Y := H(f)$ and $(Y,\mathbb{R},\pi)$ be the shift dynamical system on Y. The equation (8) (the family of equation (9)) may be written in the form (6), where $F : Y\times E\mapsto E$ is defined by equality $F(g,u) := g(0,u)$ for all $g\in H(f)=Y$ and $u\in E$, then $F(g^t,u)=g(t,u)$ ($g^t(s,u) := \sigma(t,g)(s,u) = g(t+s,u)$ for all $t,s\in\mathbb{R}$ and $u\in E$).*

2.1. **Recurrent, Almost Periodic and Almost Automorphic Motions.**

Let $(X,\mathbb{S},\pi)$ be a dynamical system.

Definition 2.8. *A number $\tau\in\mathbb{S}$ is called an $\varepsilon>0$ shift of x (respectively, almost period of x), if $\rho(x\tau,x)<\varepsilon$ (respectively, $\rho(x(\tau+t),xt)<\varepsilon$ for all $t\in\mathbb{S}$).*

Definition 2.9. *A point $x\in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon>0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x\in X$.*

Definition 2.10. *If the point $x\in X$ is almost recurrent and the set $H(x) := \overline{\{xt \mid t\in\mathbb{S}\}}$ is compact, then x is called recurrent.*

Denote by $\mathfrak{N}_x := \{\{t_n\} :\ \{t_n\}\subset\mathbb{S}$ such that $\{\pi(t_n,x)\}\to x$ as $n\to\infty\}$.

Definition 2.11. *A point $x\in X$ of the dynamical system $(X,\mathbb{S},\pi)$ is called Levitan almost periodic* [20]*, if there exists a dynamical system $(Y,\mathbb{S},\sigma)$ and a Bohr almost periodic point $y\in Y$ such that $\mathfrak{N}_y\subseteq\mathfrak{N}_x$.*

Definition 2.12. *A point $x\in X$ is called stable in the sense of Lagrange (st.L), if its trajectory $\Sigma_x := \Phi\{\pi(t,x)\ :\ t\in\mathbb{S}\}$ is relatively compact.*

Definition 2.13. *A point $x\in X$ is called almost automorphic in the dynamical system $(X,\mathbb{S},\pi)$, if the following conditions hold:*

(i) *x is st.L;*
(ii) *the point $x\in X$ is Levitan almost periodic.*

Lemma 2.14. [9] *Let $(X,\mathbb{S},\pi)$ and $(Y,\mathbb{S},\sigma)$ be two dynamical systems, $x\in X$ and the following conditions be fulfilled:*

(i) *a point $y\in Y$ is Levitan almost periodic;*
(ii) $\mathfrak{N}_y\subseteq\mathfrak{N}_x$.

Then the point x is Levitan almost periodic, too.

Corollary 2.15. *Let $x\in X$ be a st.L point, $y\in Y$ be an almost automorphic point and $\mathfrak{N}_y\subseteq\mathfrak{N}_x$. Then the point x is almost automorphic too.*

Proof. Let y be an almost automorphic point, then by Lemma 2.14 the point $x\in X$ is Levitan almost periodic. Since x is st.L, then it is almost automorphic. ☐

Remark 2.16. *We note (see, for example,* [20] *and* [28]*) that if $y \in Y$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$, then the point $x = h(y)$ is a stationary (τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{T}_1, \pi)$.*

Definition 2.17. *A point $x_0 \in X$ is called* [28, 30]

- *pseudo recurrent if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ and $p \in \Sigma_{x_0}$ there exist numbers $L = L(\varepsilon, t_0) > 0$ and $\tau = \tau(\varepsilon, t_0, p) \in [t_0, t_0 + L]$ such that $\tau \in \mathfrak{T}(p, \varepsilon)$);*
- *pseudo periodic (or uniformly Poisson stable) if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ there exists a number $\tau = \tau(\varepsilon, t_0) > t_0$ such that $\tau \in \mathfrak{T}(p, \varepsilon)$) for any $p \in \Sigma_{x_0}$.*

Remark 2.18. *1. Every pseudo periodic point is pseudo recurrent.*

2. If $x \in X$ is pseudo recurrent, then

- *it is Poisson stable;*
- *every point $p \in H(x)$ is pseudo recurrent;*
- *there exist pseudo recurrent points for which the set $H(x_0)$ is compact but not minimal* [26, ChV]*;*
- *there exist pseudo recurrent points which are not almost automorphic (respectively, pseudo periodic)* [26, ChV].

2.2. **Comparability of Motions by the Character of Recurrence.** In this subsection following B. A. Shcherbakov [27, 28] (see also [3], [4, ChI]) we introduce the notion of comparability of motions of dynamical system by the character of their recurrence. While studying stable in the sense of Poisson motions this notion plays the very important role (see, for example, [26, 28]).

Let $(X, \mathbb{S}, \pi)$ and $(Y, \mathbb{S}, \sigma)$ be dynamical systems, $x \in X$ and $y \in Y$. Denote by $\Sigma_x := \{\pi(t, x) : \ t \in \mathbb{S}\}$ and $\mathfrak{M}_x := \{\{t_n\} :$ such that $\{\pi(t_n, x)\}$ converges as $n \to \infty\}$.

Definition 2.19. *A point $x_0 \in X$ is called*

a. *comparable by the character of recurrence with $y_0 \in Y$ if there exists a continuous mapping $h : \Sigma_{y_0} \mapsto \Sigma_{x_0}$ satisfying the condition*

$$h(\sigma(t, y_0)) = \pi(t, x_0) \quad \text{for any } t \in \mathbb{R}; \tag{10}$$

b. *strongly comparable by the character of recurrence with $y_0 \in Y$ if there exists a continuous mapping $h : H(y_0) \mapsto H(x_0)$ satisfying the condition*

$$h(y_0) = x_0 \text{ and } h(\sigma(t, y)) = \pi(t, h(x)) \text{ for any } y \in H(x_0) \text{ and } t \in \mathbb{R}; \tag{11}$$

c. *uniformly comparable by the character of recurrence with $y_0 \in Y$ if there exists a uniformly continuous mapping $h : \Sigma_{y_0} \mapsto \Sigma_{x_0}$ satisfying condition (10).*

Theorem 2.20. *Let $x_0 \in X$ be uniformly comparable by the character of recurrence with $y_0 \in Y$. If the spaces X and Y are complete, then x_0 is strongly comparable by the character of recurrence with $y_0 \in Y$.*

Proof. Let $h:\Sigma_{y_0}\mapsto \Sigma_{x_0}$ be a uniformly continuous mapping satisfying condition (10) and the spaces X and Y be complete. Then h admits a unique continuous extension $h: H(y_0)\to H(x_0)$. Now we will show that this map possesses property (11). Tho this end we note that by condition h satisfies equality (10). Let now $y\in H(y_0)$ and $t\in\mathbb{R}$, then there exists a sequence $\{t_n\}\subset\mathbb{T}$ such that $\sigma(t_n,y_0)\to y$ as $n\to\infty$ and, consequently, $\sigma(t+t_n,y_0)\to\sigma(t,y)$. Since the sequence $\{\sigma(t_n,y_0)\}$ is convergent and the map $h:\Sigma_{y_0}\mapsto\Sigma_{x_0}$ is uniformly continuous, satisfies (10) and the spaces X and Y are complete, then the sequence $\{\pi(t_n,x_0)\}=\{h(\sigma(t_n,y_0))\}$ is also convergent. Denote by $x:=\lim\limits_{n\to\infty}\pi(t_n,x_0)$. Then we have

$$h(\sigma(t,y))=\lim_{n\to\infty}h(\sigma(t_n+t,y_0))=\lim_{n\to\infty}\pi(t_n+t,x_0)=$$
$$\lim_{n\to\infty}\pi(t,\pi(t_n,x_0))=\pi(t,x)=\pi(t,h(y)).$$

Theorem is proved. □

Corollary 2.21. *The uniform comparability implies strong comparability (if the phase spaces are complete) and strong comparability implies the (simple) comparability.*

Theorem 2.22. [3],[4, ChI] *Let X and Y be two complete metric spaces, then the following statement are equivalent:*

(i) *the point x_0 is strongly comparable by the character of recurrence with $y_0\in Y$;*
(ii) $\mathfrak{M}_{y_0}\subseteq\mathfrak{M}_{x_0}$.

Theorem 2.23. [27] *If the spaces X and Y are complete and y_0 is Lagrange stable, then the strong comparability implies uniform comparability and, consequently, they are equivalent.*

Remark 2.24. *From Theorems* 2.20 *and* 2.23 *follows that the strong comparability of the point x_0 with y_0 is equivalent to their uniform comparability if the point y_0 is st. L and the phase space X and Y are complete. In general case these notions are apparently different* (*though we do not know the according example*).

Theorem 2.25. [26, 28] *Let $x_0\in X$ be uniformly comparable by the character of recurrence with $y_0\in Y$. If $y_0\in Y$ is pseudo recurrent (respectively, pseudo periodic), then x_0 is so.*

Proof. If y_0 is pseudo recurrent (respectively, pseudo periodic) then for any $\varepsilon>0$, $t_0\in\mathbb{T}$ and $p\in\Sigma_{x_0}$ there exist $L=L(\varepsilon,t_0)>0$ and $\tau=\tau(\varepsilon,t_0,p)\in[t_0,t_0+L]$ (respectively, there exists $\tau=\tau(\varepsilon,t_0)>t_0$) such that $\tau\in\mathfrak{T}(p,\varepsilon)$ for any $p\in\Sigma_{x_0}$. Since x_0 is uniformly comparable by the character of recurrence with y_0, then there exists a uniformly continuous mapping $h:\Sigma_{y_0}\mapsto\Sigma_{x_0}$ satisfying (10). Let $p\in\Sigma_{x_0}$, $q\in h^{-1}(p)$, $\delta=\delta(\varepsilon)>0$ be chosen from the uniform continuity of h and $\tilde{L}(\varepsilon,t_0):=L(\delta(\varepsilon),t_0)>0$ and $\tilde{\tau}(\varepsilon,t_0,p):=\tau(\delta(\varepsilon),t_0,q)\in[t_0,t_0+\tilde{L}(\varepsilon,t_0)]$ (respectively, $\tilde{\tau}(\varepsilon,t_0):=\tau(\delta(\varepsilon),t_0)>t_0$), then $\tau\in\mathfrak{T}(p,\varepsilon)$ because $\tau\in\mathfrak{T}(q,\delta)$ and $h(q)=p$. Theorem is proved. □

Theorem 2.26. *Let $\langle(X,\mathbb{T},\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a nonautonomous dynamical system and $x_0\in X$ be a conditionally Lagrange stable point (i.e., the set Σ_{x_0} is conditionally precompact), then the following statement hold:*

(i) *if* $H(x_0)\bigcap X_{y_0}$ *consists a single point* $\{x_0\}$*, where* $y_0 := h(x_0)$*, then* $\mathfrak{N}_{y_0} \subseteq \mathfrak{N}_{x_0}$*;*

(ii) *if the set* $H(x_0)\bigcap X_q$ *contains at most one point for any* $q \in H(y_0)$*, then* $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_{x_0}$*.*

Proof. Let $\{t_n\} \in \mathfrak{N}_{y_0}$, then $\sigma(t_n, y_0) \to y_0$ as $n \to \infty$. Since Σ_{x_0} is conditionally precompact and $\{\pi(t_n, x_0)\} = \Sigma_{x_0} \bigcap h^{-1}(\{\sigma(t_n, y_0)\})$, then $\{\pi(t_n, x_0)\}$ is a precompact sequence. Tho show that $\{t_n\} \in \mathfrak{N}_{x_0}$ it is sufficient to prove that the sequence $\{\pi(t_n, x_0)\}$ has at most one limiting point. Let p_i $(i = 1, 2)$ be two limiting points of $\{\pi(t_n, x_0)\}$, then there are $\{t_{k_n^i}\} \subseteq \{t_n\}$ such that $p_i := \lim\limits_{n\to\infty} \pi(t_{k_n^i}, x_0)$ $(i = 1, 2)$. Since $\{t_{k_n^i}\} \in \mathfrak{N}_{y_0}$, then $p_i \in H(x_0)\bigcap X_{y_0} = \{x_0\}$ $(i = 1, 2)$ and, consequently, $p_1 = p_2 = x_0$. Thus we have $\mathfrak{N}_{y_0} \subseteq \mathfrak{N}_{x_0}$.

Let now $\{t_n\} \in \mathfrak{M}_{y_0}$, $q \in H(y_0)$ such that $q = \lim\limits_{n\to\infty} \sigma(t_n, y_0)$ and $H(x_0)\bigcap X_q$ contains at most one point. By the same arguments as above the sequence $\{\pi(t_n, x_0)\}$ is precompact. To show that the sequence $\{\pi(t_n, x_0)\}$ converges we will use the similar reasoning as above. Let p_i $(i = 1, 2)$ be two limiting points of $\{\pi(t_n, x_0)\}$, then there are $\{t_{k_n^i}\} \subseteq \{t_n\}$ such that $p_i := \lim\limits_{n\to\infty} \pi(t_{k_n^i}, x_0)$ $(i = 1, 2)$. Since $\sigma(t_{k_n^i}, y_0) \to q$ as $n \to \infty$, then $p_i \in H(x_0)\bigcap X_q$ $(i = 1, 2)$ and, consequently, $p_1 = p_2$. Thus we have $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_{x_0}$. Theorem is completely proved. □

Theorem 2.27. *Let* $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ *be a nonautonomous dynamical system and* $x_0 \in X$ *be a conditionally Lagrange stable point. Suppose that the following conditions are fulfilled:*

a. Y *is minimal;*
b. $H(x_0)\bigcap X_{y_0}$ *consists a single point* $\{x_0\}$*, where* $y_0 := h(x_0)$*;*
c. *the set* $H(x_0)$ *is distal, i.e.,* $\inf\limits_{t\in\mathbb{T}} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$ *for any* $x_1, x_2 \in H(x_0)$ *with* $x_1 \neq x_2$ *and* $h(x_1) = h(x_2)$*.*

Then $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_{x_0}$*.*

Proof. By Theorem 2.26 to prove this statement it is sufficient to show that $H(x_0)\bigcap X_q$ contains at most one point for any $q \in H(y_0)$. If we suppose that it is not true, then there are points $q_0 \in H(y_0)$ and $p_0^i \in H(x_0)\bigcap X_{q_0}$ such that $p_0^1 \neq p_0^2$. Then by condition c. there exists a number $\alpha = \alpha(p_0^1, p_0^2) > 0$ such that

$$\rho(\pi(t, p_0^1), \pi(t, p_0^2)) \geq \alpha \tag{12}$$

for any $t \in \mathbb{T}$. Since Y is minimal, then $H(q_0) = H(y_0) = Y$ and, consequently, there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $\sigma(t_n, q_0) \to y_0$ as $n \to \infty$. Consider the sequences $\{\pi(t_n, p_0^i)\}$ $(i = 1, 2)$. Since Σ_{x_0} is conditionally precompact, then by the same arguments as in Theorem 2.26 the sequences $\{\pi(tc_n, p_0^i)\}$ $(i = 1, 2)$ are precompact too. Without loss of generality we may suppose that they are convergent. Denote by $x^i := \lim\limits_{n\to\infty} \pi(t_n, p_0^i)$ $(i = 1, 2)$. Then $x^i \in H(x_0)\bigcap X_{y_0} = \{x_0\}$ $(i = 1, 2)$ and, consequently, $x^1 = x^2 = x_0$. By the other hand according to inequality (12) we have $\rho(x^1, x^2) \geq \alpha > 0$. The obtained contradiction show that our assumption is falls, i.e., under the conditions of Theorem $H(x_0)\bigcap X_q$ contains at most one point. Theorem is proved. □

Remark 2.28. *Note that Theorems 2.26 and 2.27 coincide with the results of B. A. Shcherbakov* [26, ChIII] *(see also* [28, ChIII]*) when the point x_0 is Lagrange stable.*

3. Birkhoff's theorem for non-autonomous dynamical systems (NDS)

Let X, Y be two complete metric spaces and

$$\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle \tag{13}$$

be a non-autonomous dynamical system.

Definition 3.1. *A subset M of X is said to be a minimal set of NDS (13) if it possesses the following properties:*

a. *$h(M) = Y$;*
b. *M is positively invariant, i.e., $\pi(t,M) \subseteq M$ for any $t \in \mathbb{T}_1$;*
c. *M is a minimal subset of X possessing properties a. and b..*

Remark 3.2. *1. In the case of autonomous dynamical systems (i.e., Y consists a single point) the definition above coincides with the usual notion of minimality.*

2. If the NDS $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is periodic (i.e., there exists a τ-periodic point $y_0 \in Y$ such that $Y = \{\sigma(t,y_0) :\ t \in [0,\tau)\}$), then the nonempty compact set $M \subset X$ is a minimal set of NDS (13) if and only if the set $M_{y_0} := X_{y_0} \bigcap M$ is a minimal set of the discrete (autonomous) dynamical system (X_{y_0}, P) generated by positive powers of the map $P := \pi(\tau,\cdot) : X_{y_0} \to X_{y_0}$.

Lemma 3.3. *Let $M \subset X$ be a nonempty, closed and positively invariant subset of NDS (13) such that $h(M) = Y$, then the following statements hold:*

(i) *if $H(x) = M$ for any $x \in M$, where $H(x) := \overline{\{\pi(t,x) :\ t \in \mathbb{T}_1\}}$, then M is a minimal set of NDS (13);*
(ii) *if*
  (a) *$\mathbb{T}_1 = \mathbb{T}_2$;*
  (b) *Y is a minimal set of autonomous dynamical system $(Y,\mathbb{T}_2,\sigma)$;*
  (c) *M is a minimal subset of NDS (13) and it is conditionally compact,*
  then $H(x) = M$ for any $x \in M$.

Proof. Let $M \subset X$ be a nonempty, closed and positively invariant subset of NDS (13) such that $h(M) = Y$ and $H(x) = M$ for any $x \in M$. We will show that in this case M is a minimal set of NDS (13). If we suppose that it is not true, then there exists a subset $\tilde{M} \subset M$ such that $\tilde{M}$ is a nonempty, closed, positively invariant, $h(\tilde{M}) = Y$ and $\tilde{M} \neq M$. Let $x \in \tilde{M} \subset M$, then by condition of Lemma we have $M = H(x) \subseteq \tilde{M} \subset M$ and, consequently, $M = \tilde{M}$. The obtained contradiction proves our statement.

Suppose that M is a minimal subset of NDS (13) and it is conditionally compact. We will establish that, then $H(x) = M$ for any $x \in M$. In fact. If it is not so, then there exists a point $x_0 \in M$ such that $H(x_0) \subset M$ and $H(x_0) \neq M$. Since $h(\pi(t,x_0)) = \sigma(t,y_0)$ (where $y_0 := h(x_0)$) for any $t \in \mathbb{T}_1$ and Y is minimal, then for any $y \in Y$ there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $\sigma(t_n,y_0) \to y$ as $n \to \infty$. Since the set M is conditionally compact without loss of generality we can suppose that the sequence $\{\pi(t_n, x_0)\}$ converges. Denote by $x := \lim_{n\to\infty} \pi(t_n, x_0)$, then we have $h(x) = y$. Since $y \in Y$ is an arbitrary point, then $h(H(x_0)) = Y$. Thus we have a nonempty, positively invariant subset $H(x_0) \subset M$ such that $h(H(x_0)) = Y$ and $H(x_0) \neq M$. This fact contradicts to the minimality of M. The obtained contradiction proves the second statement of Lemma. □

Corollary 3.4. *Let $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h)$ be a non-autonomous dynamical system and $M \subset X$ be a nonempty, conditionally compact and positively invariant set. If the dynamical system $(Y, \mathbb{T}, \sigma)$ is minimal, then the subset M is a minimal subset of NDS (13) if and only if $H(x) = M$ for any $x \in M$.*

Theorem 3.5. *Suppose that $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h)$ is a non-autonomous dynamical system and X is conditionally compact, then there exists a minimal subset M.*

Proof. Denote by $\mathfrak{A}(X)$ the family of all nonempty, positively invariant and conditionally compact subsets $A \subseteq X$. Note that $\mathfrak{A}(X) \neq \emptyset$ because $X \in \mathfrak{A}(X)$. It is clear that the family $\mathfrak{A}(X)$ partially ordered with respect to inclusion $\subseteq$. Namely: $A_1 \leq A_2$ if and only if $A_1 \subseteq A_2$ for all $A_1, A_2 \in \mathfrak{A}(X)$. If $\mathcal{A} \subseteq \mathfrak{A}(X)$ is a linear ordered subfamily of $\mathfrak{A}(X)$, then the intersection M of subsets of the family $\mathcal{A}$ is nonempty. In fact. For any $y \in Y$ the family of subsets $\mathcal{A}_y := \{A_y : \ A \in \mathcal{A}\}$, where $A_y := A \bigcap X_y$, is linear ordered and, consequently,

$$M_y = \bigcap \{A_y : \ A \in \mathcal{A}\} \neq \emptyset$$

because X_y is compact. Thus M is a closed, positively invariant set such that $h(M) = Y$ and, consequently, $M \in \mathfrak{A}(X)$. By Lemma of Zorn the family $\mathfrak{A}(X)$ contains at least one minimal element M. It is clear that M is a minimal set. Theorem is proved. □

If X is a compact metric space, then X^X denote the collection of all maps from X to itself, provided with the product topology, or, what is the same thing, the topology of pointwise convergence. By Tikhonov theorem, X^X is compact.

X^X has a semigroup structure defined by the composition of maps.

Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and $y \in Y$ be a Poisson stable point. Denote by

$$E_y^+ := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}_y^{+\infty} \quad \text{such that} \quad \pi^{t_n}|_{X_y} \to \xi\},$$

where $X_y := \{x \in X | \quad h(x) = y\}$ and $\to$ means the pointwise convergence and $\mathfrak{N}_x^{+\infty} := \{\{t_n\} \in \mathfrak{N}_x : \ t_n \to +\infty \text{ as } n \to \infty\}$.

Lemma 3.6. [6, ChIX,XV] *Let $y \in Y$ be a Poisson stable point, $\langle (X, \mathbb{T}_1, \pi)$, $(Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system and X be a conditionally compact set. Then E_y^+ is a nonempty compact sub-semigroup of the semigroup $X_y^{X_y}$.*

Lemma 3.7. [6, ChIX,XV] *Let X be a conditionally compact metric space and $\langle (X, \mathbb{T}_1, \pi)$, $(Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $y \in Y$ is Poisson stable;*

(ii) $\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all* $x_1, x_2 \in X_y := h^{-1}(y) = \{x \in X : h(x) = y\}$.

Then there exists a unique point $x_y \in X_y$ *such that* $\xi(x_y) = x_y$ *for all* $\xi \in E_y^+$.

Remark 3.8. *1. If a point* $x \in X$ *is compatible by the character of the recurrence with* $y \in Y$ *and* y *is a stationary (respectively,* τ*-periodic, recurrent, Poisson stable) point, then the point* x *is so* [28].

2. If a point $x \in X$ *is strongly compatible by the character of the recurrence with* $y \in Y$ *and* y *is a stationary (respectively,* τ*-periodic, quasi periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, almost recurrent, strongly Poisson stable and* $H(y)$ *is minimal, Poisson stable) point, then the point* x *is so* [28].

Corollary 3.9. *Let* X *be a conditionally compact metric space and* $\langle (X,\mathbb{T}_1,\pi), (Y,\mathbb{T}_2,\sigma), h\rangle$ *be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point* $y \in Y$ *is Poisson stable;*
(ii) $\lim_{t\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all* $x_1, x_2 \in X_y := h^{-1}(y) = \{x \in X : h(x) = y\}$.

Then there exists a unique point $x_y \in X_y$ *which is compatible by the character of the recurrence with* $y \in Y$ *point* $x_y \in X_y$ *such that*

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,x_y)) = 0$$

for all $x \in X_y$.

Corollary 3.10. *Let* $y \in Y$ *be a stationary* (*respectively,* τ*-periodic, recurrent, Poisson stable*) *point. Then under the conditions of Corollary* 3.9 *there exists a unique stationary* (*respectively,* τ*-periodic, recurrent, Poisson stable*) *point* $x_y \in X_y$ *such that*

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,x_y)) = 0$$

for all $x \in X_y$.

Let $x_0 \in X$. Denote by $\omega_{x_0} := \bigcap_{t\ge 0} \overline{\bigcup\{\pi(\tau,x_0) :\ \tau \ge t\}}$ the ω-limit set of x_0.

Lemma 3.11. *Let* $\langle (X,\mathbb{T}_+,\pi), (Y,\mathbb{T},\sigma)\ , h\rangle$ *be a nonautonomous dynamical system and* $\Sigma_{x_0}^+ := \{\pi(t,x_0) :\ t \ge 0\}$ *be conditionally precompact. Then for any* $x \in \omega_{x_0}$ *there exists at least one entire trajectory of dynamical system* $(X,\mathbb{T}_+,\pi)$ *passing through the point* x *for* $t = 0$ *and* $\gamma(\mathbb{T}) \subseteq \omega_{x_0}$ $(\gamma(\mathbb{T}) := \{\gamma(t) |\ t \in \mathbb{T}\})$.

Proof. Let $x \in \omega_{x_0}$, then there are $\{t_n\} \subset \mathbb{T}$ such that $x = \lim_{n\to\infty} \pi(t_n,x_0)$ and $t_n \to +\infty$ as $n \to \infty$. We consider the sequence $\{\gamma_n\} \subset C(\mathbb{T},X)$ defined by equality

$$\gamma_n(t) = \pi(t+t_n,x_0), \quad \text{if} \quad t \ge -t_n \quad \text{and} \quad \gamma_n(t) = x_0 \quad \text{for} \quad t \le -t_n.$$

Let l be an arbitrary positive number. We will prove that the sequence $\{\gamma_n\}$ is equicontinuous on segment $[-l,l] \subset \mathbb{T}$. If we suppose that it is not true, then there exist $\varepsilon_0, l_0 > 0, t_n^i \in [-l_0, l_0]$ and $\delta_n \to 0 \quad (\delta_n > 0)$ such that

$$|t_n^1 - t_n^2| \le \delta_n \quad \text{and} \quad \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) \ge \varepsilon_0. \tag{14}$$

We may suppose that $t_n^i \to t_0 \quad (i = 1, 2)$. Since $t_n \to +\infty$ as $n \to \infty$, then there exists a number $n_0 \in \mathbb{N}$ such that $t_n \ge l$ for any $n \ge n_0$. From (14) we obtain

$$\varepsilon_0 \le \rho(\gamma_n(t_n^1), \gamma_n(t_n^2)) = \rho(\pi(t_n^1 + l_0, \pi(t_n - l_0, x_0)), \pi(t_n^2 + l_0, \pi(t_n - l_0, x_0))) \tag{15}$$

for any $n \ge n_0$. Note that $h(\pi(t_n - l_0, x_0)) = \sigma(t_n - l_0, y_0) \to \sigma(y_0, -l_0)$ as $n \to \infty$. Since $\Sigma_{x_0}^+$ is conditionally precompact, then the sequence $\{\pi(t_n - l_0, x_0)\}$ is relatively compact. Without loss of generality we can suppose that $\{\pi(t_n - l_0, x_n)\}$ converges and denote by $\bar{x}$ its limit. Passing into limit in (15) as $n \to \infty$ and taking into account above we obtain

$$\varepsilon_0 \le \rho(\pi(t_0 + l_0, \bar{x}), \pi(t_0 + l_0, \bar{x})) = 0.$$

The obtained contradiction proves our statement.

Now we will prove that the set $\{\gamma_n(t) : \ t \in [-l, l], \ n \in \mathbb{N}\}$ is precompact. To this end we note that for any $n \ge n_0$ we have $h(\gamma_n(t)) = h(\pi(t + t_n, x_0)) = \sigma(t, \sigma(t_n, y_0))$ and, consequently, the set $K := \{\sigma(t, \sigma(t_n, y_0)) : \ t \in [-l, l]\} \subset Y$ is precompact. Since the set $\Sigma_{x_0}^+$ is conditionally precompact and $\{\gamma_n(t) : \ t \in [-l, l]\} \subset h^{-1}(K) \bigcap \Sigma_{x_0}^+$, then the set $\{\gamma_n(t) : \ t \in [-l, l]\}$ is also precompact and, consequently, $\{\gamma_n\}$ is a relatively compact sequence of $C(\mathbb{T}, X)$ since $\{\gamma_n\}$ is equicontinuous on $[-l, l]$.

Let γ be a limiting point of the sequence $\{\gamma_n\}$, then there exists a subsequence $\{\gamma_{k_n}\}$ such that $\gamma(t) = \lim_{n\to\infty} \gamma_{k_n}(t)$ uniformly on every segment $[-l, l] \subset \mathbb{T}$. In particular $\gamma \in C(\mathbb{T}, X)$ and $\gamma(t) \in \omega_{x_0}$ for any $t \in \mathbb{T}$ because $\gamma(t) = \lim_{n\to\infty} \pi(t + t_n, x_0)$. We note that

$$\pi^t \gamma(s) = \lim_{n\to\infty} \pi^t \gamma_{k_n}(s) = \lim_{n\to\infty} \gamma_{k_n}(s + t) = \gamma(s + t)$$

for all $t \in \mathbb{T}_+$ and $s \in \mathbb{T}$. Finally, we see that $\gamma(0) = \lim_{n\to\infty} \gamma_{k_n}(0) = \lim_{n\to\infty} \pi^{t_{k_n}} x_0 = x$, i.e., γ is an entire trajectory of dynamical system $(X, \mathbb{T}_+, \pi)$ passing through point x. The Lemma is completely proved. □

4. Semi-group Dynamical Systems

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \lambda), h \rangle$ (respectively, $\langle W, \varphi, (Y, \mathbb{T}, \lambda) \rangle$, where $\varphi : \mathbb{T}_+ \times W \times Y \mapsto W$) be a semi-group non-autonomous dynamical system (respectively, a semi-group cocycle), where $\mathbb{T}_+ := \{t \in \mathbb{T} \mid t \ge 0\}$.

A continuous mapping $\gamma : \mathbb{T} \mapsto X$ (respectively, $\nu : \mathbb{T} \mapsto W$) is called an entire trajectory of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ (respectively, semi-group cocycle $\langle W, \varphi, (Y, \mathbb{T}, \lambda) \rangle$ or shortly φ) passing through the point x (respectively, (u, y)), if $\gamma(0) = x$ (respectively, $\nu(0) = u$) and $\pi(t, \gamma(s)) = \gamma(t + s)$ (respectively, $\varphi(t, \nu(s), \lambda(s, y)) = \nu(t + s)$) for all $t \in \mathbb{T}_+$ and $s \in \mathbb{T}$.

The entire trajectory γ of the semigroup dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be comparable with $y \in Y$ by the character of recurrence ($(Y, \mathbb{T}, \lambda)$ is a two-sided dynamical system) if $\mathfrak{N}_y \subseteq \mathfrak{N}_\gamma$, where $\mathfrak{N}_\gamma := \{\{t_n\} \subset \mathbb{R} \mid$ such that the sequence $\{\gamma(t+t_n)\}$ converges uniformly with respect to t on every compact from $\mathbb{T}$, i.e. it converges in the space $C(\mathbb{T}, X)$.

Remark 4.1. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \lambda), h\rangle$ be a semi-group non-autonomous dynamical system, M be subset of X. Denote by $\tilde{M} := \{x \in M \mid$ there exists at least one entire trajectory γ of $(X, \mathbb{T}_+, \pi)$ passing through the point x with condition $\gamma(\mathbb{T}) \subseteq M\}$. It is easy to see that the set $\tilde{M}$ is invariant, i.e. $\pi(t, \tilde{M}) = \tilde{M}$ for all $t \in \mathbb{T}_+$. Moreover, $\tilde{M}$ is the maximal invariant set which is contained in M.*

Denote by $\Phi(M)$ the family of all entire trajectories γ of a semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with condition $\gamma(\mathbb{T}) \subseteq M$.

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \lambda), h\rangle$ be a semi-group non-autonomous dynamical system, $x_0 \in X$, $y_0 := h(x_0)$ and $y \in \omega_{y_0}$. Denote by Φ_y the family of all full trajectory γ of semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ satisfying the condition: $h(\gamma(0)) = y$ and $\gamma(\mathbb{T}) \subseteq \omega_{x_0}$.

Lemma 4.2. [2] *Assume that $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \lambda), h\rangle$ is a semi-group non-autonomous dynamical system, M is a conditionally compact subset of X, and $\tilde{M} \neq \emptyset$. Then, the following statements hold:*

- (i) *the set $\tilde{M}$ is closed;*
- (ii) *$\tilde{Y} := h(\tilde{M})$ is a closed and invariant subset of $(Y, \mathbb{T}, \lambda)$;*
- (iii) *$\Phi(M)$ is a closed and shift invariant subset of $C(\mathbb{T}, M)$ and, consequently, on $\Phi(M)$ is induced a shift dynamical system $(\Phi(M), \mathbb{T}, \sigma)$ by Bebutov's dynamical system $(C(\mathbb{T}, M), \mathbb{T}, \sigma)$;*
- (iv) *the mapping $H : \Phi(M) \mapsto \tilde{Y}$ defined by equality $H(\gamma) := h(\gamma(0))$ is a homomorphism of the dynamical system $(\Phi(M), \mathbb{T}, \sigma)$ onto $(\tilde{Y}, \mathbb{T}, \lambda)$, i.e. the map H is continuous and*
$$H(\sigma(t, \gamma)) = \lambda(t, H(\gamma))$$
for all $\gamma \in \Phi(M)$ and $t \in \mathbb{T}$;
- (v) *the set $\Phi(M)$ is conditionally compact with respect to $(\Phi(M), \tilde{Y}, H)$;*
- (vi) *if $\gamma_1, \gamma_2 \in \Phi(M)$ and $h(\gamma_1(0)) = h(\gamma_2(0))$, then the following conditions are equivalent:*
  - a. $\lim\limits_{t \to +\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$, *where ρ is the distance on X;*
  - b. $\lim\limits_{t \to +\infty} d(\sigma(t, \gamma_1), \sigma(t, \gamma_2)) = 0$, *where d is the Bebutov's distance on $\Phi(M)$.*

Theorem 4.3. [2] *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \lambda), h\rangle$ be a semi-group non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

- a. *there exists a point $x_0 \in X$ such that $H^+(x_0)$ is conditionally compact;*
- b. *the point $y_0 := h(x_0) \in Y$ is Poisson stable;*
- c.
$$\lim_{t \to +\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$
for all entire trajectories γ_1 and γ_2 of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ with the conditions: $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $h(\gamma_1(0)) = h(\gamma_2(0)) = y_0$.

Then, there exists a unique entire trajectory $\gamma \in \Phi_{y_0}$ of $(X, \mathbb{T}_+, \pi)$ possessing the following properties:

(i) $\gamma(\mathbb{T}) \subseteq H^+(x_0)$;
(ii) $h(\gamma(0)) = y_0$;
(iii) γ *is comparable with* $y_0 \in Y$ *by the character of recurrence.*

Corollary 4.4. [2] *Assume the conditions of Theorem 4.3 hold and that*

$$\lim_{t\to+\infty} \rho(\gamma_1(t), \gamma_2(t)) = 0$$

for all entire trajectories γ_1 *and* γ_2 *of the semi-group dynamical system* $(X, \mathbb{T}_+, \pi)$ *with the conditions:* $\gamma_i(\mathbb{T})$ *(i=1,2) is conditionally compact and* $h(\gamma_1(0)) = h(\gamma_2(0)) = y_0$.

Then, there exists a unique entire trajectory γ *of* $(X, \mathbb{T}_+, \pi)$, *which is comparable with* $y_0 \in Y$ *by the character of recurrence, and which satisfies the following properties:*

(i) $\gamma(\mathbb{T})$ *is conditionally compact;*
(ii) $h(\gamma(0)) = y_0$.

Corollary 4.5. [2] *Let* $y_0 \in Y$ *be a stationary (τ-periodic, almost automorphic, almost recurrent, Levitan almost periodic, Poisson stable) point. Then under the conditions of Theorem 4.3 there exists a unique stationary (τ-periodic, almost automorphic, almost recurrent, Levitan almost periodic, Poisson stable) entire trajectory* γ *of dynamical system* $(X, \mathbb{T}_+, \pi)$ *such that* $\gamma(\mathbb{T}) \subseteq H^+(x_0)$.

Remark 4.6. *Theorem 4.3 and Corollaries 4.4 and 4.5 remain true if we replace condition (16) by equality:*

$$\lim_{n\to\infty} \rho(\gamma_1(t_n), \gamma_2(t_n)) = 0$$

for any $\gamma_1, \gamma_2 \in \Phi_{y_0}$ *and* $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$.

Theorem 4.7. [2] *Let* $\langle (X, \mathbb{T}_+, \pi),\ (Y, \mathbb{T}, \lambda), h\rangle$ *be a semi-group non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

a. *there exists a point* $x_0 \in X$ *such that* $H^+(x_0)$ *is conditionally compact;*
b. *the point* $y_0 := h(x_0) \in Y$ *is Poisson stable;*
c.
$$\lim_{t\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$$
for any $x_1, x_2 \in X_{y_0}$.

Then, there exists a unique entire trajectory $\gamma \in \Phi_{y_0}$ *of* $(X, \mathbb{T}_+, \pi)$ *possessing the following properties:*

(i) $\gamma(\mathbb{T}) \subseteq H^+(x_0)$;
(ii) $h(\gamma(0)) = y_0$;
(iii) γ *is comparable with* $y_0 \in Y$ *by the character of recurrence and*

$$\lim_{t\to+\infty} \rho(\pi(t, x), \gamma(t)) = 0 \tag{16}$$

for any $x \in X_{y_0}$.

Corollary 4.8. [2] *Let $y_0 \in Y$ be a stationary (τ-periodic, almost automorphic, almost recurrent, Levitan almost periodic, Poisson stable) point. Then under the conditions of Theorem 4.7 there exists a unique stationary (τ-periodic, almost automorphic, almost recurrent, Levitan almost periodic, Poisson stable) entire trajectory γ of dynamical system $(X, \mathbb{T}_+, \pi)$ such that $\gamma(\mathbb{T}) \subseteq H^+(x_0)$ and equality (16) takes place.*

The entire trajectory γ of the semi-group dynamical system $(X, \mathbb{T}_+, \pi)$ is said to be strongly comparable with the point y of group dynamical system $(Y, \mathbb{T}, \lambda)$ by the character of recurrence, if $\mathfrak{M}_y \subseteq \mathfrak{M}_\gamma$, where $\mathfrak{M}_\gamma := \{\{t_n\} \subset \mathbb{T} \mid$ the sequence $\sigma(t_n, \gamma)$ converges in the space $C(\mathbb{T}, X)\}$.

A point $x \in X$ is said to be strongly Poisson stable if each point $p \in H(x)$ is Poisson stable.

Theorem 4.9. *Let X be a conditionally compact metric space and $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *the dynamical system $(Y, \mathbb{T}, \sigma)$ is minimal;*
(ii) *the point $y \in Y$ is strongly Poisson stable;*
(iii)

$$\lim_{t \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0 \tag{17}$$

for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.

Then there exists a unique point $x_y \in X_y$ which is strongly compatible by the character of the recurrence with $y \in Y$ and

$$\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, x_y)) = 0$$

for any $x \in X_y$.

Proof. By Lemma 3.7 there exists a unique fixed point $\tilde{x}_y \in X_y$ of the semigroup E_y^+. By Corollary 3.9 the point $\tilde{x}_y$ is a unique point in $\tilde{M}$ comparable by character of recurrence with the point y. Let $\tilde{M} := \overline{\{\pi(t, x_y)\ : t \in \mathbb{T}_1\}}$ then it is a conditionally compact positively invariant set and taking into account the minimality of Y using the same argument as in the proof of Lemma 3.3 we have $H(\tilde{M}) = Y$. By Theorem 3.5 there exists a minimal set $M \subset \tilde{M}$. By Corollary 3.9 there exists a point $x_y \in M$ which is a unique point in M comparable by character of recurrence with the point y and, consequently, x_y coincides with the point $\tilde{x}_y$. We will show that $M_q := M \cap X_q$ (for all $q \in H(y) := \overline{\{\sigma(t, y)\ : t \in \mathbb{T}\}}$) consists a single point. If we suppose that it is not true then there exist $q_0 \in H(y)$ and $x_1, x_2 \in M_{q_0}$ such that $x_1 \neq x_2$. By Corollary 3.9 there exists a unique point $x_{q_0} \in M_{q_0}$ which is compatible by the character of recurrence with the point q_0. Without loss of generality we may suppose that $x_{q_0} = x_1$. Since the set M is minimal, then there exists a sequence $\{t_n\} \in \mathfrak{N}_{q_0}^{+\infty}$ such that $\{\pi(t_n, x_1)\} \to x_2$. On the other hand taking into consideration the inclusion $\mathfrak{N}_{q_0} \subseteq \mathfrak{N}_{x_1}$ we have $\{\pi(t_n, x_1)\} \to x_1$ and, consequently, $x_1 = x_2$. The obtained contradiction prove our statement. Now to finish the proof of Theorem it is sufficient to apply Theorem 2.26. □

Corollary 4.10. *Let $y \in Y$ be a stationary (respectively, τ–periodic, almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set) point. Then under the conditions of Theorem 4.9 there exists a unique stationary (respectively, τ–periodic, almost periodic, recurrent, strongly Poisson stable and $H(y_0)$ is a minimal set) point $x_y \in X_y$ such that*

$$\lim_{t\to+\infty} \rho(\pi(t,x),\pi(t,x_y)) = 0$$

for all $x \in X_y$.

Proof. This statement directly follows from Theorem 4.9 and Remark 3.8. □

Theorem 4.11. *Let $\langle (X,\mathbb{T}_+,\pi),\ (Y,\mathbb{T},\lambda),h\rangle$ be a semi-group non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

a) *there exists a point $x_0 \in X$ such that $H^+(x_0)$ is conditionally compact;*
b) *the point $y_0 := h(x_0) \in Y$ is strongly Poisson stable;*
c) *the set $H(y_0)$ is minimal;*
d)

$$\lim_{t\to+\infty} \rho(\gamma_1(t),\gamma_2(t)) = 0 \tag{18}$$

for all entire trajectories γ_1 and γ_2 of the semi-group dynamical system $(X,\mathbb{T}_+,\pi)$ with the conditions: $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $h(\gamma_1(0)) = h(\gamma_2(0))$.

Then there exists a unique entire trajectory γ of $(X,\mathbb{T}_+,\pi)$ possessing the following properties:

(i) *$\gamma(\mathbb{T}) \subseteq H^+(x_0)$;*
(ii) *$h(\gamma(0)) = y_0$;*
(iii) *γ is strongly comparable by the character of recurrence with $y_0 \in Y$.*

Proof. Let $M := H^+(x_0)$. Then, by Lemma 3.11 we have that $\Phi(M) \neq \emptyset$. Consider the group non-autonomous dynamical system $\langle (\Phi(M),\mathbb{T},\sigma),(\tilde{Y},\mathbb{T},\lambda),H\rangle$ (see Lemma 4.2). By Lemma 3.11 the point y_0 belongs to $\tilde{Y}$. According to Lemma 4.2 all conditions of Theorem 4.9 are fulfilled and, consequently, we obtain the existence of at least one entire trajectory γ of the dynamical system $(X,\mathbb{T}_+,\pi)$ which is strongly comparable with $y_0 \in Y$ by the character of recurrence, and $\gamma(\mathbb{T}) \subseteq H^+(x_0)$. To finish the proof it is sufficient to show that there exists at most one entire trajectory of $(X,\mathbb{T}_+,\pi)$ with the properties (i)-(iii). Let γ_1 and γ_2 be two entire trajectories satisfying (i)-(iii). In particular, $\gamma_i(\mathbb{T}) \subseteq H^+(x_0)$ and $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_{\gamma_i}$ $(i=1,2)$. Then we also have $\mathfrak{N}_{y_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i=1,2)$. From assumption c) we obtain

$$\lim_{t\to+\infty} d(\sigma(t,\gamma_1),\sigma(t,\gamma_2)) = 0.$$

On the other hand, there exists a sequence $\{t_n\} \in \mathfrak{N}_{y_0} \subseteq \mathfrak{N}_{\gamma_i}$ $(i=1,2)$ such that $t_n \to +\infty$ and, consequently,

$$d(\gamma_1,\gamma_2) = \lim_{n\to+\infty} d(\sigma(t_n,\gamma_1),\sigma(t_n,\gamma_2)) = 0,$$

i.e., $\gamma_1 = \gamma_2$, and the proof is completed. □

Corollary 4.12. *In addition to assumptions in Theorem 4.11, suppose that*

$$\lim_{t\to+\infty}\rho(\gamma_1(t),\gamma_2(t))=0$$

for all entire trajectories γ_1 *and* γ_2 *of the semi-group dynamical system* $(X,\mathbb{T}_+,\pi)$ *with the conditions:* $\gamma_i(\mathbb{T})$ *(i=1,2) is conditionally compact and* $h(\gamma_1(0))=h(\gamma_2(0))$.

Then there exists a unique entire trajectory $\gamma\in\Phi_{y_0}$ *of* $(X,\mathbb{T}_+,\pi)$*, which is strongly comparable with* $y_0\in Y$ *by the character of recurrence, and such that* $\gamma(\mathbb{T})$ *is conditionally precompact and*

$$\lim_{t\to+\infty}\rho(\pi(t,x),\gamma(t))=0$$

for any $x\in X_{y_0}$.

Proof. This statement follows by a slight modification of the proof of Theorem 4.11. □

Corollary 4.13. *Let* $y_0\in Y$ *be a stationary (respectively,* τ*-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(y_0)$ *is a minimal set) point. Then under the conditions of Theorem 4.11 ,there exists a unique stationary (respectively,* τ*-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(y_0)$ *is a minimal set) entire trajectory* γ *of the dynamical system* $(X,\mathbb{T}_+,\pi)$ *such that* $\gamma(\mathbb{T})\subseteq H^+(x_0)$.

Proof. This statement follows easily from Theorem 4.11 and Remarks 3.8 (item 2). □

Corollary 4.14. *Under the conditions of Corollary 4.13 if*

$$\lim_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$$

for any $x_1,x_2\in H^+(x_0)$ *with* $h(x_1)=h(x_2)$*. Then there exists a unique full trajectory* $\gamma\in\Phi_{y_0}$ $(y_0:=h(x_0))$ *which is strongly comparable by character of recurrence with the point* y_0 *and*

$$\lim_{t\to+\infty}\rho(\pi(t,x),\gamma(t))=0$$

for any $x\in X_{y_0}$.

Remark 4.15. *Theorems 4.9 and 4.11 remain true if we replace condition (17) (respectively, (18)) by equality:*

$$\lim_{n\to\infty}\rho(\gamma_1(t_n),\gamma_2(t_n))=0 \tag{19}$$

for any $\gamma_1,\gamma_2\in\Phi_y$, $\{t_n\}\in\mathfrak{N}_y^{+\infty}$ *and* $y\in H(y_0)$.

5. Bohr/Levitan almost periodic, almost automorphic and Poisson stable functions

Let (X,ρ) be a compete metric space. Denote by $C(\mathbb{R},X)$ the family of all continuous functions $f:\mathbb{R}\mapsto X$ equipped with the distance

$$d(f,g):=\sup_{l>0}d_l(f,g),$$

where $d_l(f,g) := \min\{\max_{|t|\le l}\rho(f(t),g(t)); l^{-1}\}$. The metric d is complete and it defines on $C(\mathbb{R},X)$ the compact-open topology. Let $h \in \mathbb{R}$ denote by f^h the h-translation of f, that is, $f^h(s) := f(s+h)$ for all $s \in \mathbb{R}$.

Let us recall the types of Poisson stable functions to be studied in this paper; we refer the reader to [24, 26, 28, 30] and the references therein.

Definition 5.1. A function $\varphi \in C(\mathbb{R},X)$ is called *stationary* (respectively, τ-*periodic*) if $\varphi(t) = \varphi(0)$ (respectively, $\varphi(t+\tau) = \varphi(t)$) for all $t \in \mathbb{R}$.

Definition 5.2. Let $\varepsilon > 0$. A number $\tau \in \mathbb{R}$ is called ε-*almost period* of the function φ if $\rho(\varphi(t+\tau),\varphi(t)) < \varepsilon$ for any $t \in \mathbb{R}$.

Denote by $\mathcal{T}(\varphi,\varepsilon) := \{\tau \in \mathbb{R} : \ \rho(\varphi(t+\tau),\varphi(t)) < \varepsilon \text{ for any } t \in \mathbb{R}\}$ the set of ε-almost periods of φ.

Definition 5.3. A function $\varphi \in C(\mathbb{R},X)$ is said to be *Bohr almost periodic* if the set of ε-almost periods of φ is *relatively dense* for each $\varepsilon > 0$, i.e., for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that $\mathcal{T}(\varphi,\varepsilon) \cap [a, a+l] \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 5.4. A function $\varphi \in C(\mathbb{R},X)$ is said to be *pseudo-periodic* in the positive (respectively, negative) direction if for each $\varepsilon > 0$ and $l > 0$ there exists a ε-almost period $\tau > l$ (respectively, $\tau < -l$) of the function φ. The function φ is called pseudo-periodic if it is pseudo-periodic in both directions.

Remark 5.5. A function $\varphi \in C(\mathbb{R},X)$ is pseudo-periodic in the positive (respectively, negative) direction if and only if there is a sequence $t_n \to +\infty$ (respectively, $t_n \to -\infty$) such that φ^{t_n} converges to φ uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

Definition 5.6. The *hull of* φ, denoted by $H(\varphi)$, is the set of all the limits of φ^{h_n} in $C(\mathbb{R},X)$, i.e.

$$H(\varphi) := \{\psi \in C(\mathbb{R},X) : \psi = \lim_{n\to\infty} \varphi^{h_n} \text{ for some sequence } \{h_n\} \subset \mathbb{R}\}.$$

Definition 5.7. A number $\tau \in \mathbb{R}$ is said to be ε-*shift* for $\varphi \in C(\mathbb{R},X)$ if $d(\varphi^\tau,\varphi) < \varepsilon$.

Denote by $\mathfrak{T}(\varepsilon,\varphi) := \{\tau \in \mathbb{R} : \ \rho(\varphi^\tau,\varphi) < \varepsilon\}$ the set of all ε-shifts of φ.

Definition 5.8. A function $\varphi \in C(\mathbb{R},X)$ is called *almost recurrent (in the sense of Bebutov)* if for every $\varepsilon > 0$ the set $\mathfrak{T}(\varphi,\varepsilon)$ is relatively dense.

Definition 5.9. A function $\varphi \in C(\mathbb{R},X)$ is called *Lagrange stable* if $\{\varphi^h : \ h \in \mathbb{R}\}$ is a precompact subset of $C(\mathbb{R},X)$.

Definition 5.10. A function $\varphi \in C(\mathbb{R},X)$ is called *Birkhoff recurrent* if it is almost recurrent and Lagrange stable.

Definition 5.11. A function $\varphi \in C(\mathbb{R},X)$ is called:

- *Poisson stable* in the positive (respectively, negative) direction if for every $\varepsilon > 0$ and $l > 0$ there exists $\tau > l$ (respectively, $\tau < -l$) such that $d(\varphi^\tau,\varphi) < \varepsilon$. The function φ is called Poisson stable if it is Poisson stable in both directions;
- *strongly Poisson stable* if every function $p \in H(\varphi)$ is Poisson stable.

In what follows, we denote as well Y a complete metric space.

Definition 5.12. A function $\varphi \in C(\mathbb{R}, X)$ is called *Levitan almost periodic* if there exists a Bohr almost periodic function $\psi \in C(\mathbb{R}, Y)$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{T}(\psi, \delta) \subseteq \mathfrak{T}(\varphi, \varepsilon)$.

Remark 5.13.
(i) Every Bohr almost periodic function is Levitan almost periodic.
(ii) The function $\varphi \in C(\mathbb{R}, \mathbb{R})$ defined by equality $\varphi(t) = \dfrac{1}{2 + \cos t + \cos \sqrt{2}t}$ is Levitan almost periodic, but it is not Bohr almost periodic [20, ChIV].

Definition 5.14. A function $\varphi \in C(\mathbb{R}, X)$ is said to be *Bohr almost automorphic* if it is Levitan almost periodic and Lagrange stable.

Definition 5.15. A function $\varphi \in C(\mathbb{R}, X)$ is called *quasi-periodic with the spectrum of frequencies* $\nu_1, \nu_2, \dots, \nu_k$ if the following conditions are fulfilled:

(i) the numbers $\nu_1, \nu_2, \dots, \nu_k$ are rationally independent;
(ii) there exists a continuous function $\Phi : \mathbb{R}^k \to X$ such that $\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$ for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;
(iii) $\varphi(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

Let $\varphi \in C(\mathbb{R}, X)$. Denote by $\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) the family of all sequences $\{t_n\} \subset \mathbb{R}$ such that $\varphi^{t_n} \to \varphi$ (respectively, $\{\varphi^{t_n}\}$ converges) in $C(\mathbb{R}, X)$ as $n \to \infty$.

By $\mathfrak{N}^u_\varphi$ (respectively, $\mathfrak{M}^u_\varphi$) we denote the family of sequences $\{t_n\} \in \mathfrak{N}_\varphi$ such that φ^{t_n} converges to φ (respectively, φ^{t_n} converges) uniformly in $t \in \mathbb{R}$ as $n \to \infty$.

Remark 5.16.
(i) The function $\varphi \in C(\mathbb{R}, X)$ is pseudo-periodic in the positive (respectively, negative) direction if and only if there is a sequence $\{t_n\} \in \mathfrak{N}^u_\varphi$ such that $t_n \to +\infty$ (respectively, $t_n \to -\infty$) as $n \to \infty$.
(ii) Let $\varphi \in C(\mathbb{R}, X)$, $\psi \in C(\mathbb{R}, Y)$ and $\mathfrak{N}^u_\psi \subseteq \mathfrak{N}^u_\varphi$. If the function ψ is pseudo-periodic in the positive (respectively, negative) direction, then so is φ.

Definition 5.17. A function $\varphi \in C(\mathbb{R}, X)$ is called [25, 30] *pseudo-recurrent* if for any $\varepsilon > 0$ and $l \in \mathbb{R}$ there exists $L = L(\varepsilon, l) > 0$ such that for any $\tau_0 \in \mathbb{R}$ we can find a number $\tau = \tau(\varepsilon, l, t_0) \in [l, l + L]$ satisfying

$$\sup_{|t| \le 1/\varepsilon} \rho(\varphi(t + \tau_0 + \tau), \varphi(t + \tau_0)) \le \varepsilon.$$

Remark 5.18. ([25, 30])

(i) Every Birkhoff recurrent function is pseudo-recurrent, but the inverse statement is not true in general.
(ii) If the function $\varphi \in C(\mathbb{R}, X)$ is pseudo-recurrent, then every function $\psi \in H(\varphi)$ is pseudo-recurrent.
(iii) If the function $\varphi \in C(\mathbb{R}, X)$ is Lagrange stable and every function $\psi \in H(\varphi)$ is Poisson stable, then φ is pseudo-recurrent.

6. Linear systems

6.1. **Linear non-autonomous dynamical systems with exponential dichotomy.** Let $(E, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ (or shortly φ)

be a linear cocycle over dynamical system $(Y, \mathbb{T}, \sigma)$ with the fiber E, i.e., φ is a continuous mapping from $\mathbb{T} \times E \times Y$ into E satisfying the following conditions:

(i) $\varphi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
(ii) $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_+$, $u \in E$ and $y \in Y$;
(iii) for all $(t, y) \in \mathbb{T}_+ \times Y$ the mapping $\varphi(t, \cdot, y) : E \mapsto E$ is linear.

Denote by $[E]$ the Banach space of all linear bounded operators A acting on the space E equipped with the operator norm $||A|| := \sup\limits_{|x| \le 1} |Ax|$.

Example 6.1. *Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space E (for example $\Lambda = \{A_0 + B | B \in [E]\}$, where A_0 is a closed operator that acts on E). Consider the following linear differential equation*

$$x' = A(\sigma(t, y))x, \quad (y \in Y) \tag{20}$$

where $A \in C(Y, \Lambda)$. We assume that the following conditions are fulfilled for equation (20):

a. *for any $u \in E$ and $y \in Y$ equation (20) has exactly one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, u, y) = u$;*
b. *the mapping $\varphi : (t, u, y) \to \varphi(t, u, y)$ is continuous in the topology of $\mathbb{R}_+ \times E \times Y$.*

Under the above assumptions the equation (20) generates a linear cocycle $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber E.

Example 6.2. *Let Λ be some complete metric space of linear closed operators acting into Banach space E. Consider the differential equation*

$$x' = A(t)x, \tag{21}$$

where $A \in C(\mathbb{R}, \Lambda)$. Along this equation (21) consider its H-class, i.e., the following family of equations

$$x' = B(t)x, \tag{22}$$

where $B \in H(A)$. We assume that the following conditions are fulfilled for equation (21) and its H-class (22):

a. *for any $u \in E$ and $B \in H(A)$ equation (22) has exactly one mild solution $\varphi(t, u, B)$ (i.e. $\varphi(\cdot, u, B)$ is continuous, defined on $\mathbb{R}_+$ and satisfies of equation*
$$\varphi(t, v, B) = U(t, B)v + \int_0^t U(t - \tau, B^\tau)\varphi(\tau, v, B)d\tau$$
and the condition $\varphi(0, u, B) = v$);
b. *the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$.*

Denote by $(H(A), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A)$. Under the above assumptions the equation (21) generates a linear cocycle $\langle E, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fiber E.

Note that equation (21) and its H-class can be written in the form (20). In fact. We put $Y := H(A)$ and denote by $\mathcal{A} \in C(Y, \Lambda)$ defined by equality $\mathcal{A}(B) := B(0)$ for all $B \in H(A) = Y$, then $B(\tau) = \mathcal{A}(\sigma(B, \tau)$ $(\sigma(\tau, B) := B^{\tau}$, where $B^{\tau}(t) := B(t + \tau)$ for all $t \in \mathbb{R}$). Thus the equation (21) with its H-class can be rewrite as follow

$$x' = \mathcal{A}(\sigma(t, B))x. \ (B \in H(A))$$

We will consider example of partial differential equations which satisfy the above conditions a.-b.

Example 6.3. Consider the differential equation

$$u' = (A_1 + A_2(t))u, \tag{23}$$

where A_1 is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $A_2 \in C(\mathbb{R}, [E])$. The results of [17], [20] imply that equation (23) satisfies conditions a.-b. from Example 6.21.

Definition 6.4. *Recall (see, for example,* [10, Ch.VI]*) that the linear cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is hyperbolic (or equivalently, satisfies the condition of exponential dichotomy), if there exists a continuous projection valued function $P : Y \to [E]$ satisfying:*

(i) *$P(\sigma(t, y))U(t, y) = U(t, y)P(y)$ for all $(t, y) \in \mathbb{T} \times Y$:*
(ii) *for all $(t, y) \in \mathbb{T} \times Y$ the operator $U_Q(t, y)$ is invertible as an operator from $ImQ(y)$ to $ImQ(\sigma(t, y))$, where $Q(y) := Id_E - P(y)$ and $U_Q(t, y) := U(t, y)Q(y)$;*
(iii) *there exist constants $\nu > 0$ and $\mathcal{N} > 0$ such that*

$$\|U_P(t, y)\| \le \mathcal{N} e^{-\nu t} \text{ and } \|U_Q(t, y)^{-1}\| \le \mathcal{N} e^{-\nu t} \tag{24}$$

for all $y \in Y$ and $t \in \mathbb{S}_+$, where $U_P(t, y) := U(t, y)P(y)$ and $U(t, y) = \varphi(t, \cdot, y)$.

Lemma 6.5. *Suppose that the linear cocycle $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is hyperbolic and $\gamma \in C(\mathbb{R}, E)$ is a full trajectory of cocycle φ, i.e., there exists a point $y_0 \in Y$ such that $\gamma(t) = U(t - \tau, \sigma(\tau, y_0))\gamma(\tau)$ for any $t \ge \tau$ and $\tau \in \mathbb{R}$. If*

$$\sup\{|\gamma(t)|;\ t \in \mathbb{R}\} < +\infty \ ,$$

then $\gamma(\tau) = 0$ for any $\tau \in \mathbb{R}$.

Proof. Note that

$$\gamma(t) = U(t - \tau, \sigma(\tau, y_0))\gamma(\tau) = U(t - \tau, \sigma(\tau, y_0))P(\sigma(\tau, y_0))\gamma(\tau) +$$

$$U(t - \tau, \sigma(\tau, y_0))Q(\sigma(\tau, y_0))\gamma(\tau)$$

for any $t \ge \tau$ and, consequently, $C_1 := \sup\{|U(t - \tau, \sigma(\tau, y_0))Q(\sigma(\tau, y))\gamma(\tau)| :\ t \ge \tau\} < +\infty$. According to (24) we have

$$C_1 \ge |U(t - \tau, \sigma(\tau, y_0))Q(\sigma(\tau, y))\gamma(\tau)| \ge \mathcal{N} e^{\nu(t - \tau)} |Q(\sigma(\tau, y_0))|$$

for any $t \ge \tau$ and, consequently, $Q(\sigma(\tau, y_0))\gamma(\tau) = 0$ for any $\tau \in \mathbb{R}$. This means that $\gamma(\tau) = P(\sigma(\tau, y_0))\gamma(\tau)$ for any $\tau \in \mathbb{R}$.

On the other hand $\gamma(t) = U(t-\tau, \sigma(\tau, y_0))\gamma(\tau) = U(t-\tau, \sigma(\tau, y_0))Q(\sigma(\tau, y_0))\gamma(\tau)$ for any $t \ge \tau$. Taking into account (24) we obtain

$$|\gamma(t)| \le \mathcal{N} e^{-\nu(t-\tau)} C \tag{25}$$

for any $t \ge \tau$ and $\tau \in \mathbb{R}$, where $C := \sup\{|\gamma(\tau)| : \ \tau \in \mathbb{R}\}$. Passing into limit in (25) as $\tau \to -\infty$ we obtain $\gamma(t) = 0$ for any $t \in \mathbb{R}$. Lemma is proved. □

6.2. **Relationship between different definitions of hyperbolicity.** Along with classical definition of hyperbolicity (Definition 6.4) we will use an other definition given below. And in this Subsection we establish the relation between two definitions of hyperbolicity for linear homogeneous differential equations with continuous (bounded) coefficients.

Let $A \in C(\mathbb{R}, \Lambda)$, $\Sigma_A := \{A^\tau : \ A^\tau(t) := A(t+\tau) \text{ for all } t \in \mathbb{R}\}$ and $H(A) := \overline{\Sigma}_A$, where by bar is denoted the closure of the set Σ_A in $C(\mathbb{R}, \lambda)$. In this Subsection we will suppose that the operator-function $A \in C(\mathbb{R}, \Lambda)$ is regular, i.e., for all $B \in H(A)$ there exists a unique solution $\varphi(t, u, B)$ of equation

$$x' = B(t)x \quad (B \in \Sigma_A) \tag{26}$$

with initial data $\varphi(0, u, B) = u$ defined on $\mathbb{R}_+$.

Definition 6.6. *(Classical definition* [12, 13]*) Let $A \in C(\mathbb{R}, E)$. Linear differential equation*

$$x' = A(t)x \tag{27}$$

satisfies the exponential dichotomy if there exist a projection $P(A) : E \to E$ and the positive constants $\mathcal{N}$ and $\nu > 0$ such that

$$\begin{gathered} ||U(t, A)P(A)U^{-1}(\tau, A)|| \le \mathcal{N} e^{-\nu(t-\tau)} \text{ for any } t > \tau \\ \text{and} \\ ||U(t, A)(I - P(A))U^{-1}(\tau, A)|| \le \mathcal{N} e^{\nu(t-\tau)} \text{ for any } t < \tau. \end{gathered} \tag{28}$$

Lemma 6.7. [4, ChIII] *Suppose that equation (27) satisfies the exponential dichotomy, $A \in C(\mathbb{R}, [E])$ and $B \in H(A)$. Then equation (26) also satisfies the exponential dichotomy.*

Definition 6.8. *Differential equation (27) is said to be hyperbolic (satisfies the condition of exponential dichotomy), if there are two projections: $P, Q : H(A) \mapsto [E]$ ($P^2(B) = P(B)$ and $Q^2(B) = Q(B)$ for all $B \in H(A)$) such that*

(i) *the mappings P and Q are continuous;*
(ii) *$P(B) + Q(B) = Id_E$ for all $B \in H(A)$;*
(iii) *$U(t, B)P(B) = P(B^t)U(t, B)$ for all $t \in \mathbb{R}_+$ and $B \in H(A)$, where $U(t, B) := \varphi(t, \cdot, B)$;*
(iv) *the mapping $U_Q(t, B) := U(t, B)Q(B) : Im(Q(B) \mapsto Im(Q(B))$ is invertible;*
(v) *there are positive numbers $\mathcal{N}$ and ν such that $||U_P(t, B)|| \le \mathcal{N} e^{-\nu t}$ and $||[U_Q(t, B)]^{-1}|| \le \mathcal{N} e^{-\nu t}$ for any $t \ge 0$, where $U_P(t, B) := U(t, B)P(B)$ for any $t \in \mathbb{R}_+$.*

Remark 6.9. *Note that the definition above of the hyperbolicity means that the cocycle* $\langle E, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ *generated by equation (27) is hyperbolic in the sense of Definition 6.4.*

Lemma 6.10. *Let* $A \in C(\mathbb{R}, [E])$. *If (27) is hyperbolic in the sense of Definition 6.8, then it is also so in the classical sense.*

Proof. Let (27) be hyperbolic in the sense of Definition 6.8, $P(A) : H(A) \mapsto [E]$ projection-operator and $\mathcal{N}, \nu$ positive constants which figure in definition of hyperbolicity.

Denote by $P(s) := U(s, A)P(A)U^{-1}(\tau, A)$ and $Q(s) := Id_E - P(s)$. It easy to check that $P^2(s) = P(s)$ for any $s \in \mathbb{R}$. Since equation (27) is hyperbolic (in the sense of Definition 6.8), then

$$||U(t, A^s)P(s)|| \le \mathcal{N}e^{-\nu t} \tag{29}$$

for any $(t, s) \in \mathbb{R}_+ \times \mathbb{R}$ and

$$||U(t, A^s)Q(s)|| \le \mathcal{N}e^{-\nu t} \tag{30}$$

for any $(t, s) \in \mathbb{R}_+ \times \mathbb{R}0$, where $Q(A^s) := Id_E - P(A^s)$.

Note that

$$U(t, A)P(A)U^{-1}(\tau, A) = U(t - \tau, A^\tau)U(\tau, A)P(A)U^{-1}(\tau, A) = U(t - \tau, A^\tau)P(s) \tag{31}$$

for any $t > \tau$ and

$$U(t, A)Q(A)U^{-1}(\tau, A) = U(t - \tau, A^\tau)U(\tau, A)Q(A)U^{-1}(\tau, A) = U(t - \tau, A^\tau)Q(s) \tag{32}$$

for any $t < \tau$.

From (29)- (32) it follows that

$$||U(t, A)P(A)U^{-1}(\tau, A)|| \le \mathcal{N}e^{-\nu(t-\tau)} \tag{33}$$

and

$$||U(t, A)Q(A)U^{-1}(\tau, A)|| \le \mathcal{N}e^{-\nu(t-\tau)} \tag{34}$$

for any $t > \tau$, because $U(t, A)Q(A)U^{-1}(\tau, A) = (U_Q(t, \tau))^{-1}$, where $U_Q(t, \tau) = U(t - \tau, A^\tau)Q(A^\tau)$. Lemma is proved. □

Lemma 6.11. *Let* E *be a finite-dimensional Banach space. Then, if equation (27) is hyperbolic (in the sense of Definition 6.6), then it is also hyperbolic in the sense of Definition 6.8.*

Proof. Let $A \in C(\mathbb{R}, [E])$. Denote by Σ_A the set of all translations of A, i.e., $\Sigma_A = \{A^s : s \in \mathbb{R}\}$ and $A^s(t) := A(t + s)$ for all $t \in \mathbb{R}$. Suppose that equation (27) is hyperbolic with projection $P(A)$ and the constants $\mathcal{N} > 0$ and $\nu > 0$ which figure in Definition 6.6. Let $B \in \Sigma_A$, then there is a number $s \in \mathbb{R}$ such that $B = A^s$. We put $P(B) := U(s, A)P(A)U^{-1}(s, A)$, then it easy to check that $P(B)$ is a projection (i.e., $P^2(B) = P(B)$). Note that

$$U(t, A^s)P(A^s)U(\tau, A^s) = U(t + s, A)P(A)U^{-1}(\tau + s, A) \tag{35}$$

and

$$U(t, A^s)Q(A^s)U(\tau, A^s) = U(t + s, A)Q(A)U^{-1}(\tau + s, A) \tag{36}$$

for all $t, \tau, s \in \mathbb{R}$, where $Q(A^s) := Id_E - P(A^s)$. From (28), (35) and (36) it follows that

$$||U(t, B)P(B)U^{-1}(\tau, B)|| \le \mathcal{N}e^{-\nu(t-\tau)} \text{ for all } t > \tau \tag{37}$$

and

$$||U(t, B)Q(B)U^{-1}(\tau, B)|| \le \mathcal{N}e^{-\nu(t-\tau)} \text{ for all } t < \tau, \tag{38}$$

for all $B \in \Sigma_A$. Denote by $H(A) := \overline{\{A^s : s \in \mathbb{R}\}}$, where by bar is denoted the closure in the space $C(\mathbb{R}, [E])$. will show that the mapping $P : \Sigma_A \mapsto [E]$ admits a unique extension $P : H(A) \mapsto [E]$ possessing the following properties:

(i) the mapping $P : H(A) \mapsto [E]$ is continuous;
(ii) $P(B^\tau) = U(\tau, B)P(B)U^{-1}(\tau, B)$ for all $\tau \in \mathbb{R}$ and $B \in H(A)$.

Let $B \in H(A)$, then there exists a sequence $\{\tau_n\} \subset \mathbb{R}$ such that $B = \lim_{n\to\infty} A^{\tau_n}$. Consider the sequence $\{P(A^{\tau_n}\} \subset [E]$ (respectively $\{Q(A^{\tau_n}\} \subset [E]$). Under the conditions of Lemma 6.11 the sequence $\{P(A^{\tau_n}\}$ (respectively $\{Q(A^{\tau_n}\})$ is relatively compact in $[E]$. Now we will establish that the sequence $\{P(A^{\tau_n}\}$ (respectively $\{Q(A^{\tau_n}\}$ admits at most one limiting point. In fact, if P' (respectively Q') is a limiting point of $\{P(A^{\tau_n})\}$ then there exists a subsequence $\{\tau_n'\} \subseteq \{\tau_n\}$ such that $P' = \lim_{n\to\infty} P(A^{\tau_n'})$, $Q' = \lim_{n\to\infty} Q(A^{\tau_n'})$, $P'^2 = P'$ (respectively, $Q'^2 = Q'$) and $P' + Q' = I_E$. By Lemma 6.7 the equation

$$y' = B(t)y \tag{39}$$

is hyperbolic with the projections $P(B)$ and $Q(B)$ and theses projections are defined uniquely. From the last fact we obtain that $P' = P(B)$ (respectively, $Q' = Q(B)$, i.e., the sequence $\{P(A^{\tau_n})\}$ (respectively, $P' = P(B)$ (respectively, $Q' = Q(B)$, i.e., the sequence $\{P(A^{\tau_n})\}$) admits a unique limiting point $P(B)$ (respectively, $Q(B)$). It is clear that the mapping $P : \Sigma_A \mapsto [E]$ (respectively, $Q : \Sigma_A \mapsto [E]$) admits a (unique) extension on $H(A)$ and it is defined by equality $P(B) := \lim_{n\to\infty} P(A^{\tau_n})$ (respectively, $P(B) := \lim_{n\to\infty} P(A^{\tau_n})$), where $\{\tau_n\}$ is a subsequence of $\mathbb{R}$ such that $B = \lim_{n\to\infty} A^{\tau_n}$).

Now we will prove that the mapping $P : H(A) \mapsto [E]$ (respectively, $Q : H(A) \mapsto [E]$) is continuous. Let $B \in H(A)$ and $\{B_k\}$ be a subset of $H(A)$ such that $d(B, B_k) \le 1/k$ for all $k \in \mathbb{N}$. Since $B_k \in H(A)$, then there exists a sequence $\{\tau_n^k\}$ such that $B_k = \lim_{n\to\infty} A_{\tau_n^k}$. Let $n_k \in \mathbb{N}$ such that

$$d(B_k, A^{\tau_n^k}) \le 1/k \ \text{ and } \ ||P(B_k) - P(A^{\tau_n^k})|| \le 1/k \tag{40}$$

for all $n \ge n_k$. Consider the sequence $\{\tau_k'\}$, where $\tau_k' := \tau_{n_k}^k$, and note that from (40) we have $\lim_{k\to\infty} A^{\tau_k'} = B$ and, consequently (see above)

$$\lim_{k\to\infty} P(A^{\tau_k'}) = P(B). \tag{41}$$

Thus we have

$$||P(B_k) - P(B)|| \le ||B_k - P(A^{\tau_k'})|| + ||P(A^{\tau_k'}) - P(B)|| \tag{42}$$

for all $k \in \mathbb{N}$. Passing to limit in (42) as $k \to \infty$ and taking in consideration (40) and (41) we obtain $P(B) = \lim_{k\to\infty} P(B_k)$, i.e., the mapping $P : H(A) \mapsto [E]$ is continuous. Analogously may be established the continuity of the mapping $Q : H(A) \mapsto [E]$.

Let $B \in H(A)$ and $\tau \in \mathbb{R}$, then there exists a sequence $\{\tau_n\} \subset \mathbb{R}$ such that $A^{\tau_n} \to B$ as $n \to \infty$ and, consequently, $U(t, A^{\tau_n}) \to U(t, B)$, $U^{-1}(\tau, B) \to U^{-1}(\tau, B)$ and $P(A^{\tau_n}) \to P(B)$ as $n \to \infty$. From the above facts we obtain

$$P(B^{\tau}) = P(\lim_{n\to\infty} A^{\tau+\tau_n}) = \lim_{n\to\infty} P(A^{\tau+\tau_n}) = \lim_{n\to\infty} U(\tau, A^{\tau_n})P(A^{\tau_n})U^{-1}(\tau, A^{\tau_n}) = U(\tau, B)P(B)U^{-1}(\tau, B). \tag{43}$$

Analogously we can prove that $Q(B^{\tau}) = U(\tau, B)Q(B)U^{-1}(\tau, B)$. Thus equation (27) is hyperbolic in the sense of Definition 6.8. Lemma is proved. □

Lemma 6.12. [5, Ch.III] *Let $A \in C(\mathbb{R}, [E])$ and $\varphi(t, u, A)$ be a unique solution of equation (27) with initial data $\varphi(0, u, A) = u$. Then the following statements hold:*

(i) *the map $A \mapsto U(\cdot, A)$ of $C(\mathbb{R}, [E])$ to $C(\mathbb{R}, [E])$ is continuous, where $U(t, A)$ is the Cauchy's operator* [13] *of equation (27) and*
(ii) *the map $(t, u, A) \mapsto \varphi(t, u, A)$ of $\mathbb{R} \times E \times C(\mathbb{R}, [E])$ to E is continuous.*

Theorem 6.13. *Let $A \in C(\mathbb{R}, [E])$ and E be finite-dimensional. Then equation (27) is hyperbolic in the sense of Definition 6.8 if and only if it is hyperbolic in the sense of Definition 6.6.*

Proof. This statement follows from Lemmas 6.10 and 6.11. □

Remark 6.14. *Theorem 6.13 (respectively, Lemma 6.11) remains true also for the infinite-dimensional equations under the following condition: $A \in C(\mathbb{R}, [E])$ and the family of projections $\{P(t)\} := \{U(t, A)P(A)U^{-1}(t, A) : \ t \in \mathbb{R}\} \subset [E]$ is precompact in $[E]$.*

Theorem 6.15. *Suppose that the following conditions are fulfilled:*

(i) *the operator-function $A \in C(\mathbb{R}, [E])$ is τ-periodic (respectively, quasi-periodic, almost periodic, almost automorphic, recurrent);*
(ii) *equation (27) is hyperbolic (satisfies the condition of exponential dichotomy).*

Then the operator-function the operator-functions $P(t) := U(t, A)P(A)U^{-1}(t, A)$ and $Q(t) := U(t, A)Q(A)U^{-1}(t, A)$ are τ-periodic (respectively, quasi-periodic, almost periodic, almost automorphic, recurrent) and $\mathfrak{M}_A \subseteq \mathfrak{M}_P$.

Proof. We will prove this statement for the operator-function $P(t)$, because the statement for the operator-function $Q(t)$ can be proved using the same arguments.

Let $A \in C(\mathbb{R}, [E])$ be τ-periodic (respectively, quasi-periodic, almost periodic, almost automorphic, recurrent), then the operator-function A is bounded on $\mathbb{R}$ and by Theorem 6.13 (see its proof) the function $\tilde{h} : \Sigma_A \mapsto [E]$, defined by equality $\tilde{h}(A^s) = P(s)$ ($\forall\ s \in \mathbb{R}$), is uniformly continuous function and, consequently, it admits a unique continuous extension on $\overline{\Sigma}_A$. Consider the mapping $h : H(A) \mapsto H(P)$ defined by $h(B) = P_B$ ($\forall\ B \in H(A)$), where $P_B : \mathbb{R} \mapsto [E]$ and

$$P_B(t) := U(t,B)P(B)U^{-1}(t,B) \ (\forall\ t \in \mathbb{R}). \tag{44}$$

Note that the map h possesses the following properties:

(i) $h(A) = P_A$;

(ii) $h(B^s) = P_B^s$ for any $s \in \mathbb{R}$, where P_B^s is the s-translation of P_B, i.e.,

$$P_B^s(t) = P_B(t+s) \tag{45}$$

for any $t \in \mathbb{R}$;

(iii) h is continuous.

In fact. The first statement is evident. To establish the second statement we note that

$$\begin{aligned} P_{B^s}(t) = U(t,B^s)P(B^s)U^{-1}(t,B^s) = \\ U(t,B^s)U(s,B)P(B)U^{-1}(s,B)U^{-1}(t,B^s) = \\ U(t+s,B)P(B)U^{-1}(t+s,B) = P_B(t+s) \end{aligned} \tag{46}$$

for any $t, s \in \mathbb{R}$. The second statement follows from (44)-(46).

Let now $B \in H(A)$ be an arbitrary point and $\{B_n\} \subset H(A)$ such that $B_n \to B$ as $n \to \infty$ (in the topology of the space $C(\mathbb{R},[E])$), then $h(B_n) = P_{B_n}$. Since $P_{B_n}(t) = U(t,B_n)P(B_n)U^1(t,B_n)$, then $h(B_n) \to h(B)$ as $n \to \infty$, because $P(B_n) \to P(B)$ (see the proof of the Theorem 6.13) and conform Lemma 6.12 we have $U(t,B_n) \to U(t,B)$ (respectively, $U^{-1}(t,B_n) \to U^{-1}(t,B)$) in the topology of the space $C(\mathbb{R},[E])$.

Under the conditions of Theorem the point A (of the shift dynamical system $(C(\mathbb{R},[E]),\mathbb{R},\sigma)$) is stable in the sense of Lagrange and the point P_A (of the shift dynamical system $(C(\mathbb{R},[E]),\mathbb{R},\sigma)$ is uniformly comparable with A by character of recurrence. Now to finish the proof of Theorem it is sufficient to apply Remark 2.16. □

Corollary 6.16. *Under the condition of Theorem 6.15, if A is almost periodic (respectively, almost automorphic), then the operator-function P is also almost periodic (respectively, almost automorphic) and its frequency module is contained in the frequency module of A.*

Remark 6.17. *Note that for finite-dimensional equations (*$\dim E < \infty$*), if*

- *the operator-function $A \in C(\mathbb{R}.[E])$ is almost periodic, then Corollary 6.16 is well known (see* [11] *and* [12]*);*
- *the operator-function $A \in C(\mathbb{R}.[E])$ is almost automorphic, then Corollary 6.16 was proved in the work* [21].

6.3. **Linear non-homogeneous (affine) dynamical systems.** Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a linear cocycle over dynamical system $(Y,\mathbb{R},\sigma)$ with the fiber E, $f \in C(Y,\mathbb{B})$

and ψ be a mapping from $\mathbb{T}\times E\times Y$ into E defined by equality

$$\psi(t,u,y):=U(t,y)u+\int_0^t U(t-\tau,\sigma(\tau,y))f(\sigma(\tau,y))d\tau \quad \text{if } \mathbb{S}=\mathbb{R} \tag{47}$$

and

$$\psi(t,u,y):=U(t,y)u+\sum_{\tau=0}^{t} U(t-\tau,\sigma(\tau,y))f(\sigma(\tau,y)) \quad \text{if } \mathbb{S}=\mathbb{Z}. \tag{48}$$

From the definition of the mapping ψ it follows that ψ possesses the following properties:

1. $\psi(0,u,y)=u$ for any $(u,y)\in E\times Y$;
2. $\psi(t+\tau,u,y)=\psi(t,\psi(\tau,u,y),\sigma(\tau,y))$ for any $t,\tau\in\mathbb{T}$ and $(u,y)\in E\times Y$;
3. the mapping $\psi:\mathbb{T}\times E\times Y\mapsto E$ is continuous;
4. $\psi(t,\lambda u+\mu v,y)=\lambda\psi(t,u,y)+\mu\psi(t,v,y)$ for any $t\in\mathbb{T}$, $u,v\in E$, $y\in Y$ and $\lambda,\mu\in\mathbb{R}$ (or $\mathbb{C}$) with condition $\lambda+\mu=1$, i.e., the mapping $\psi(t,\cdot,y): E\mapsto E$ is affine for every $(t,y)\in\mathbb{T}\times Y$.

Definition 6.18. *A triplet $\langle E,\varphi,(Y,\mathbb{S},Y)\rangle$is called an affine (non-homogeneous) cocycle over dynamical system $(Y,\mathbb{T},Y)$ with the fiber E, if the φ is a mapping from $\mathbb{T}\times E\times Y$ into E possessing the properties 1.-4.*

Remark 6.19. *If we have a linear cocycle $\langle E,\varphi,(Y,\mathbb{R},\sigma)\rangle$ over dynamical system $(Y,\mathbb{R},\sigma)$ with the fiber E and $f\in C(Y,\mathbb{B})$, then by equality (47) (respectively, by (48)) is defined an affine cocycle $\langle E,\psi,(Y,\mathbb{R},\sigma)\rangle$ over dynamical system $(Y,\mathbb{S},\sigma)$ with the fiber E which is called an affine (non-homogeneous) cocycle associated by linear cocycle φ and the function $f\in C(Y,E)$.*

Example 6.20. *Let Y be a complete metric space, $(Y,\mathbb{R},\sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space E and $f\in C(Y,E)$. Consider the following linear non-homogeneous differential equation*

$$x'=A(\sigma(t,y))x+f(\sigma(t,y)),\quad (y\in Y) \tag{49}$$

where $A\in C(Y,\Lambda)$. We assume that conditions a. and b. from Example 6.1 are fulfilled for equation (20).

Under the above assumptions equation (20) generates a linear cocycle $\langle E,\varphi,(Y,\mathbb{R},\sigma)\rangle$ over dynamical system $(Y,\mathbb{R},\sigma)$ with the fiber E. According to Remark 6.19 by equality (47) is defined a linear non-homogeneous cocycle $\langle E,\psi,(Y,\mathbb{R},\sigma)\rangle$ over dynamical system $(Y,\mathbb{R},\sigma)$ with the fiber E. Thus every non-homogeneous linear differential equations (49), under conditions a. and b. generates a linear non-homogeneous cocycle ψ.

Example 6.21. Let Λ be some complete metric space of linear closed operators acting into Banach space E and $f\in C(\mathbb{R},E)$. Consider a linear non-homogeneous differential equation

$$x'=A(t)x+f(t), \tag{50}$$

where $A\in C(\mathbb{R},\Lambda)$. Along this equation (50) consider its H-class, i.e., the following family of equations

$$x'=B(t)x+g(t), \tag{51}$$

where $(B, g) \in H(A, f)$. We assume that the following conditions are fulfilled for equation (50) and its H-class (51):

a. for any $u \in E$ and $B \in H(A)$ equation (51) has exactly one mild solution $\varphi(t, u, B)$ and the condition $\varphi(0, u, B) = v$;
b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$.

Denote by $(H(A, f), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A, f)$. Under the above assumptions the equation (50) generates a linear cocycle $\langle E, \varphi, (H(A, f), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A, f), \mathbb{R}, \sigma)$ with the fiber E. Denote by ψ a mapping from $\mathbb{R}_+ \times E \times H(A, f)$ into E defined by equality

$$\psi(t, u, (B, g)) := U(t, B)u + \int_0^t U(t - \tau, B^\tau)g(\tau)d\tau,$$

then ψ possesses the following properties:

(i) $\psi(0, u, (B, g)) = u$ for any $(u, (B, g)) \in E \times H(A, f)$;
(ii) $\psi(t + \tau, u, (B, g)) = \psi(t, \psi(\tau, u, (B, g)), (B^\tau, g^\tau))$ for any $t, \tau \in \mathbb{T}$ and $(u, (B, g)) \in E \times H(A, f)$;
(iii) the mapping $\psi : \mathbb{T} \times E \times H(A, f) \mapsto E$ is continuous;
(iv) $\psi(t, \lambda u + \mu v, (B, g)) = \lambda\psi(t, u, (B, g)) + \mu\psi(t, v, (B, g))$ for any $t \in \mathbb{T}$, $u, v \in E$, $(B, g) \in H(A, f)$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) with condition $\lambda + \mu = 1$, i.e., the mapping $\psi(t, \cdot, (B, g)) : E \mapsto E$ is affine for every $(t, (B, g)) \in \mathbb{T} \times H(A, f)$.

Thus, every linear non-homogeneous differential equation of the form (50) (and its H-class (51)) generates a linear non-homogeneous cocycle $\langle E, \psi, (H(A, f), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A, f), \mathbb{R}, \sigma)$ with the fiber E.

Remark 6.22. *1. If $\Lambda = [E]$, $A \in C(\mathbb{R}, [E])$ and , then according to Lemma 6.12 conditions a. and b. in Example are fulfilled. Thus equation (50) with operator function $A \in C(\mathbb{R}, [E])$ generates a linear non-homogeneous cocycle ψ.*

2. A closed linear operator $A : D(A) \to E$ with dense domain $D(A)$ is said [17] *to be* sectorial *if one can find a $\phi \in (0, \frac{\pi}{2})$, an $M \geq 1$, and a real number a such that the sector*

$$S_{a,\phi} := \{\lambda \mid |arg\ (\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ of A and $\|(\lambda I - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for any $\lambda \in S_{a,\phi}$. An important class of sectorial operators is formed by elliptic operators [17], [18].

Consider the differential equation

$$u' = (A_1 + A_2(t))u, \tag{52}$$

where A_1 is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $A_2 \in C(\mathbb{R}, [E])$.

The results of [17, 20]*, imply that equation (52) satisfies conditions (i)-(iii).*

Note that equation (50) (and its H-class (51)) can be written in the form (20). In fact. We put $Y := H(A, g)$ and denote by $\mathcal{A} \in C(H(A, f), \Lambda)$ (respectively, $\{ \in C(H(A, f), E))$ defined by equality $\mathcal{A}(B, g) := B(0)$ (respectively, $\{(B, g) = g(0))$ for any $(B, g) \in H(A, f)$, then $B(\tau) = \mathcal{A}(B^\tau, g^\tau)$ (respectively, $g(\tau) = \{(B^\tau, g^\tau))$, where $B^\tau(t) := B(t+\tau)$ and $g^\tau(t) := g(t+\tau)$ for any $t \in \mathbb{R}$). Thus the equation (50) with its H-class can be rewrite as follow

$$x' = \mathcal{A}(\sigma(t,B))x + \mathcal{F}(\sigma(t,B)).\ (B,g) \in H(A,f))$$

7. Linear Stochastic Differential Equations

Consider the linear nonhomogeneous equation

$$\dot{x} = A(t)x + f(t)$$

on the Banach space E, where $f \in C(\mathbb{R}, E)$ and $A(t)$ generates a cocycle $\langle E,\ \varphi,\ (H(A,f),\ \mathbb{R},\ \sigma)\rangle$ (or shortly φ) with fiber E and base (driving system) $(H(A,f),\ \mathbb{R},\ \sigma)$.

Denote by $C_b(\mathbb{R}, E)$ the Banach space of all continuous and bounded mappings $\varphi : \mathbb{R} \to E$ equipped with the norm $||\varphi||_\infty := \sup\{|\varphi(t)| :\ t \in \mathbb{R}\}$.

Let $(H, |\cdot|)$ be a real separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $L^2(\mathbb{P}, H)$ be the space of H-valued random variables x such that

$$\mathbb{E}|x|^2 := \int_\Omega |x|^2 d\mathbb{P} < \infty.$$

Then $L^2(\mathbb{P}, H)$ is a Hilbert space equipped with the norm

$$||x||_2 := \Big(\int_\Omega |x|^2 d\mathbb{P} \Big)^{1/2}.$$

For $f \in C_b(\mathbb{R}, L^2(\mathbb{P}, H))$, the space of bounded continuous mappings from $\mathbb{R}$ to $L^2(\mathbb{P}, H)$, we denote $||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||_2$.

Consider the following linear stochastic differential equation

$$dx(t) = (A(t)x(t) + f(t)dt + g(t)dW(t), \tag{53}$$

where $f, g \in C(\mathbb{R}, H)$, $A(t)$ is an generator of cocycle φ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{F}_t := \sigma\{W(u) : u \le t\}$.

Recall that an $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in\mathbb{R}}$ is said to be a mild solution of equation (53) defined on interval $I = (a,b) \subset \mathbb{R}$ if it satisfies the stochastic integral equation

$$x(t) = U(t-s, A^s)x(s) + \int_s^t U(t-\tau, A^\tau)f(\tau)ds + \int_s^t U(t-\tau, A^\tau)g(\tau)dW(\tau),$$

for any $t \ge s$ and each $t, s \in I$ $(t \ge s)$.

Let $\mathcal{P}(H)$ be the space of all Borel probability measures on H endowed with the β metric:

$$\beta(\mu,\nu) := \sup\left\{ \left| \int f d\mu - \int f d\nu \right| : ||f||_{BL} \le 1 \right\}, \quad \text{for } \mu, \nu \in \mathcal{P}(H),$$

where f are bounded Lipschitz continuous real-valued functions on H with the norms

$$||f||_{BL} = Lip(f) + ||f||_\infty,\ Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|},\ ||f||_\infty = \sup_{x \in H} |f(x)|.$$

Recall that a sequence $\{\mu_n\} \subset \mathcal{P}(H)$ is said to *weakly converge* to μ if $\int f d\mu_n \to \int f d\mu$ for any $f \in C_b(H)$, where $C_b(H)$ is the space of all bounded continuous real-valued functions on H. It is well-known that $(\mathcal{P}(H), \beta)$ is a separable complete metric space and that a sequence $\{\mu_n\}$ weakly converges to μ if and only if $\beta(\mu_n, \mu) \to 0$ as $n \to \infty$.

Definition 7.1. *A sequence of random variables $\{x_n\}$ is said to* converge in distribution *to the random variable x if the corresponding laws $\{\mu_n\}$ of $\{x_n\}$ weakly converge to the law μ of x.*

Example 7.2. Let Λ be some complete metric space of linear closed operators acting into Hilbert space H and $f, g \in C(\mathbb{R}, H)$. Consider a linear non-homogeneous stochastic differential equation (53), where $A \in C(\mathbb{R}, \Lambda)$. Along this equation (53) consider its H-class, i.e., the following family of equations

$$dx(t) = (\tilde{A}(t)x(t) + \tilde{f}(t))dt + \tilde{g}(t)dW(t), \tag{54}$$

where $(\tilde{A}, \tilde{f}, \tilde{g}) \in H(A, f, g) := \overline{\{(A^\tau, f^\tau . g^\tau) :\ \tau \in \mathbb{R}\}}$ and by bar is denoted the closer in product space $C(\mathbb{R}, \Lambda) \times C(\mathbb{R}, H) \times C(\mathbb{R}, H)$. We assume that the following conditions are fulfilled for equation

$$dx(t) = A(t)x(t)dt, \tag{55}$$

and its H-class

$$dx(t) = \tilde{A}(t)x(t)dt, \tag{56}$$

where $\tilde{A} \in H(A) := \overline{\{A^\tau :\ \tau \in \mathbb{R}\}}$ and by bar is denoted the closer in $C(\mathbb{R}, H)$:

a. for any $u \in H$ and $\tilde{A} \in H(A)$ equation (54) has exactly one mild solution $\varphi(t, u, \tilde{A})$ with the condition $\varphi(0, u, \tilde{A}) = u$;
b. the mapping $\varphi : (t, u, \tilde{A}) \to \varphi(t, u, \tilde{A})$ is continuous in the topology of $\mathbb{R}_+ \times H \times C(\mathbb{R}; \Lambda)$.

Denote by $(H(A, f, g), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A, f, g)$. Under the above assumptions the equation (55) generates a linear cocycle $\langle E, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fiber H.

Define the mapping

$$\Phi : \mathbb{R}_+ \times \mathcal{P}(H) \times H(A, f, g) \to \mathcal{P}(H),$$

with $\Phi(t, \mu, (\tilde{A}, \tilde{f}, \tilde{g}))$ being the law (or distribution) $\mathcal{L}(\varphi(t, x, \tilde{A}, \tilde{f}, \tilde{g}))$ of the solution $\varphi(t, x, \tilde{A}, \tilde{f}, \tilde{g})$ of equation

$$dx = (\tilde{A(t)}x + \tilde{f}(t))dt + \tilde{g}(t)dW, \quad x(0) = x, \tag{57}$$

where $\mathcal{L}(x) = \mu$ and $\Phi(0, \mu, (\tilde{A}, \tilde{f}, \tilde{g})) = \mu$.

We have the following result on Φ:

Theorem 7.3. [7],[8] *The mapping Φ is a continuous cocycle with base (driving system) $(H(A,f,g),\mathbb{R},\sigma)$ and fiber $\mathcal{P}(H)$, i.e., the mapping $\Phi : \mathbb{R}_+ \times \mathcal{P}(H) \times H(A,f,g) \to \mathcal{P}(H)$ is continuous and satisfies*

(58)
$$\Phi(0,\mu,(\tilde{A},\tilde{f},\tilde{g})) = \mu, \quad \Phi(t+\tau,\mu,(\tilde{A},\tilde{f},\tilde{g})) = \Phi(t,\Phi(\tau,\mu,(\tilde{A},\tilde{f},\tilde{g})),(\tilde{A}^\tau,\tilde{f}^\tau,\tilde{g}^\tau))$$

for any $t,\tau \geq 0$, $(\tilde{A},\tilde{f},\tilde{g}) \in H(A,f,g)$ and $\mu \in \mathcal{P}(H)$.

Proof. (1) The mapping Φ satisfies the cocycle property (58). Note that $\Phi(0,\mu,(\tilde{A},\tilde{f},\tilde{g})) = \mu$ by its definition. Under our assumptions (57) admits a unique solution which we denote by $\varphi(t,x,(\tilde{A},\tilde{f},\tilde{g},W))$ with $\varphi(0,x,(\tilde{A},\tilde{f},\tilde{g},W)) = x$. Let $\tilde{W}^\tau(t) = W(t+\tau) - W(\tau)$. Then $\tilde{W}^\tau$ is still a Brownian motion which shares the same distribution as that of W.

Define

(59)
$$\phi(t) := \varphi(t,x,(\tilde{A},\tilde{f},\tilde{g},W)), \quad \psi(t) := \varphi(t,\phi(\tau),(\tilde{A}^\tau,\tilde{f}^\tau,\tilde{g}^\tau,\tilde{W}^\tau)), \quad \eta(t) := \phi(t+\tau).$$

Then $\phi(t)$ is a solution of (57) with $\phi(0) = x$, and $\psi(t)$ is a solution of

$$dx = (\tilde{A}^\tau(t)x + \tilde{f}^\tau(t))dt + \tilde{g}^\tau(t)d\tilde{W}^\tau \tag{60}$$

with $\psi(0) = \phi(\tau) = \varphi(\tau,x,(\tilde{A},\tilde{f},\tilde{g},W))$. On the other hand, note that $\eta(t)$ is also a solution of (60) with $\eta(0) = \phi(\tau)$. So by the uniqueness of solutions of (60), we get $\eta(t) = \psi(t)$. That is

$$\varphi(t+\tau,x,(\tilde{A},\tilde{f},\tilde{g},W)) = \varphi(t,\varphi(\tau,x,(\tilde{A},\tilde{f},\tilde{g},W)),(\tilde{A}^\tau,\tilde{f}^\tau,\tilde{g}^\tau,\tilde{W}^\tau)). \tag{61}$$

Thus for any Brownian motion $\bar{W}$ and any random variable $\bar{x}$ which has the same distribution as that of x, the solution of the equation

$$dx = (\tilde{A}(t)x + \tilde{f}(t))dt + \tilde{g}(t)d\bar{W}, \quad x(0) = \bar{x}$$

admits the same law on H as $\phi(t)$ above. In other words, the law of solution is uniquely determined by the coefficients $\tilde{A},\tilde{f},\tilde{g}$ and the initial distribution μ. So we can simply denote the law of $\varphi(t,x,(\tilde{A},\tilde{f},\tilde{g},W))$ by $\Phi(t,\mu,(\tilde{A},\tilde{f},\tilde{g}))$. Therefore, it follows from (61) that the cocycle property holds for Φ:

$$\Phi(t+\tau,\mu,(\tilde{A},\tilde{f},\tilde{g})) = \Phi(t,\Phi(\tau,\mu,(\tilde{A},\tilde{f},\tilde{g})),(\tilde{A}^\tau,\tilde{f}^\tau,\tilde{g}^\tau)).$$

To prove the continuity of Φ it is sufficient to apply Proposal 3.1 (item c.) from [14]. □

Definition 7.4. *Let $\{\varphi(t)\}_{t\in\mathbb{R}}$ be a mild solution of equation* (53). *Then φ is called τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, Poisson stable)* in distribution *if the function $\phi \in C(\mathbb{R},\mathcal{P}(H))$ is τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, Poisson stable), where $\phi(t) := \mathcal{L}(\varphi(t))$ for any $t \in \mathbb{R}$ and $\mathcal{L}(\varphi(t)) \in \mathcal{P}(H)$ is the law of random variable $\varphi(t)$.*

Definition 7.5. *Suppose that linear (homogeneous) equation*

$$x' = A(t)x \tag{62}$$

generates a (linear) cocycle $\langle E, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$. We will say that linear homogeneous equation (62) (respectively, cocycle φ generated by equation (62)):

- *satisfies condition (S) if for entire relatively compact on $\mathbb{R}$ mild solution γ of this equation we have*

$$\lim_{t\to+\infty} |\gamma(t)| = 0 \tag{63}$$

(respectively, equality takes place for any relatively compact on $\mathbb{R}$ mild solution of every equation

$$x' = B(t)x, \tag{64}$$

where $B \in H(A)$);

(i) *is asymptotically stable if*

$$\lim_{t\to+\infty} |\varphi(t, x, A)| = 0 \tag{65}$$

for any $x \in E$, where $\varphi(t, x, A) := U(t, A)x$ (respectively, if equality $\lim_{t\to+\infty} |\varphi(t, x, B)| = 0$ takes place for any $(x, B) \in E \times H(A)$.

Lemma 7.6. *Cocycle φ generated by equation (62) satisfies condition (S)if one of the following conditions are fulfilled:*

(i) *the set $H(A)$ is compact and the cocycle φ generated by equation (62) is asymptotically stable;*
(ii) *equation (62) is hyperbolic.*

Proof. The first statement follows from the Theorem 2.37 [6, ChII].

The second statement follows from Lemma 6.5. □

Let $\varphi \in C(\mathbb{R}, H)$ be a solution of equation (53). Denote by

- $\mathfrak{N}^d_\varphi := \{\{t_n\} : \varphi^{t_n}(t)$ converges in distribution to $\varphi(t)\}$ uniformly with respect to t on every compact from $\mathbb{R}$;
- $\mathfrak{M}^d_\varphi := \{\{t_n\} : \varphi^{t_n}(t)$ converges in distribution $\}$ uniformly with respect to t on every compact from $\mathbb{R}$.

Definition 7.7. *A solution φ is said to be compatible in distribution if $\mathfrak{N}_{(A,f,g)} \subseteq \mathfrak{N}^d_\varphi$.*

Theorem 7.8. *Suppose that the following conditions are fulfilled:*

a. *$A \in C(\mathbb{R}, \Lambda)$ and equation (27) generates a continuous cocycle $\langle E, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ with fiber E over base $(H(A), \mathbb{R}, \sigma)$;*
b. *equation (62) satisfies to condition (S);*
c. *the function $(A, f, g) \in C(\mathbb{R}, \Lambda) \times C(\mathbb{R}, H) \times C(\mathbb{R}, H)$ is Poisson stable;*
d. *equation (53) admits a solution φ defined on $\mathbb{R}_+$ with precompact rang, i.e., the set $Q_+ := \overline{\varphi(\mathbb{R}_+)}$ is compact.*

Then equation (53) has a unique solution p defined on $\mathbb{R}$ with precompact rang which is compatible.

Proof. Under the conditions of Theorem 7.8 equation (53) generates a continuous cocycle $\langle (H), \Phi, (H(A,f,g), \mathbb{R}, \sigma) \rangle$ with the fiber $\mathcal{P}(H)$ over dynamical system $(H(A,f,g), \mathbb{R}, \sigma)$ (see Theorem 7.3). Denote by $\mu_0 := \mathcal{L}(\varphi(0))$, then $\Phi(\mathbb{R}_+, \mu_0, (A, f, g)) := \{\Phi(t, \mu_0, (A,f,g)) : t \in \mathbb{R}_+\}$ is precompact in $\mathcal{P}(H)$. Let $\gamma_1, \gamma_2 \in C(\mathbb{R}, H)$ be two arbitrary solutions of equation (53) with precompact ranges. Denote by $\gamma(t) := \gamma_1(t) - \gamma_2(t)$, then $\gamma \in C(\mathbb{R}, H)$ is a solution of equation (27) with the precompact range. Since equation (27) satisfies the condition (S), then we have equality (63). We will show that

$$\beta(\mathcal{L}(\gamma(t_n)), \mathcal{L}(\gamma_2(t_n))) \to 0 \tag{66}$$

as $n \to \infty$ for any $\{t_n\} \in \mathfrak{N}_{y_0}^{+\infty}$. If we suppose that it is not true, then there exist $\varepsilon_0 > 0$ and $\{t_n\} \in \mathfrak{N}_{y_0}^{+\infty}$ such that

$$\beta(\mathcal{L}(\gamma(t_n)), \mathcal{L}(\gamma_2(t_n))) \geq \varepsilon_0 \tag{67}$$

for any $n \in \mathbb{N}$. Since the set $\gamma_i(\mathbb{R}) \subset H$ $(i = 1, 2)$ is precompct, then without loss of generality we can suppose that the sequences $\{\gamma_i(t_n)\}$ $(i = 1, 2)$ are convergent. Denote by $\bar{x}_i := \lim\limits_{n\to\infty} \gamma_i(t_n)$. By equality (67) we obtain $\bar{x}_1 \neq \bar{x}_2$. Note that $\gamma(t_n) := \gamma_1(t_n) - \gamma_2(t_n) \to \bar{x}_1 - \bar{x}_2 \neq 0$ as $n \to \infty$. The last relation contradicts to condition (S) (see item b.). The obtained contradiction proves equality (66). To finish the proof it is sufficient to apply Theorem 4.3 and Remark 4.6. □

Corollary 7.9. *Under the conditions of Theorem 7.8 if (A, f, g) is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (53) has a unique solution p defined on $\mathbb{R}$ which is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable).*

Proof. This statement follows from Theorem 7.8 and Corollary 4.5. □

Theorem 7.10. *Under the condition of Theorem 7.8 if we replace condition (S) (see item b.) by asymptotic stability of equation (27), then equation (53) has a unique solution p defined on $\mathbb{R}$ which is compatible and* $\lim\limits_{t\to+\infty} \beta(\mathcal{L}(p(t)), \mathcal{L}(\varphi(t, x, (A, f, g))) = 0$.

Proof. This statement can be proved using the same arguments as in the proof of Theorem 7.8 but instead of Theorem 4.3 and Corollary 4.13 it is necessary to apply Theorem 4.11 and Corollary 4.13. □

Corollary 7.11. *Under the conditions of Theorem 7.10 if (A, f, g) is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then every solution of equation (53) is asymptotically τ-periodic (respectively, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable) in distribution.*

Proof. This statement follows from Theorem 7.10 and Corollary 4.8. □

Definition 7.12. *A solution φ is said to be strongly compatible in distribution if $\mathfrak{M}_{(A,f,g)} \subseteq \mathfrak{M}_{\varphi}^{d}$.*

Theorem 7.13. *Suppose that the following conditions are fulfilled:*

(i) *for any* $B \in H(A)$ *equation*

$$x' = B(t)x \tag{68}$$

satisfies to condition (S);

(i)

(ii) *the set* $H(A, f, g) \subset C(\mathbb{R}, \Lambda) \times C(\mathbb{R}, H) \times C(\mathbb{R}, H)$ *is minimal;*

(iii) *the function* $(A, f, g) \in C(\mathbb{R}, \Lambda) \times C(\mathbb{R}, H) \times C(\mathbb{R}, H)$ *is strongly Poisson stable;*

(iv) *equation (53) admits a solution* φ *defined on* $\mathbb{R}_+$ *with precompact rang.*

Then equation (53) has a unique solution p *defined on* $\mathbb{R}$ *with precompact rang which is strongly compatible in distribution.*

Proof. To prove this statement we will use the same ideas as in the proof of Theorem 7.8. Note that under the conditions of Theorem 7.13 equation (53) generates a continuous cocycle $\langle (H), \Phi, (H(A, f, g), \mathbb{R}, \sigma)\rangle$ with the fiber $\mathcal{P}(H)$ over dynamical system $(H(A, f, g), \mathbb{R}, \sigma)$ (see Theorem 7.3). Denote by $\mu_0 := \mathcal{L}(\varphi(0))$, then $\Phi(\mathbb{R}_+, \mu_0, (A, f, g)) := \{\Phi(t, \mu_0, (A, f, g)) :\ t \in \mathbb{R}_+\}$ is precompact in $\mathcal{P}(H)$. Let $\gamma_1, \gamma_2 \in C(\mathbb{R}, H)$ be two arbitrary solutions of some equation (57) (where $(\tilde{A}, \tilde{f}, \tilde{g}) \in H(A, f, g))$ with precompact ranges. Denote by $\gamma(t) := \gamma_1(t) - \gamma_2(t)$, then $\gamma \in C(\mathbb{R}, H)$ is a solution of equation

$$x' = \tilde{A}(t)x \tag{69}$$

with the precompact range. Since equation (69) satisfies the condition (S), then we have equality (63). We will show that

$$\beta(\mathcal{L}(\gamma(t_n)), \mathcal{L}(\gamma_2(t_n))) \to 0 \tag{70}$$

as $n \to \infty$ for any $\gamma_1, \gamma_2 \in \Phi_y$, $y \in H(y_0)$ and $\{t_n\} \in \mathfrak{N}_y^{+\infty}$. If we suppose that it is not true, then there exist $q \in H(y_0)$, $\varepsilon_0 > 0$ and $\{t_n\} \in \mathfrak{N}_q^{+\infty}$ such that

$$\beta(\mathcal{L}(\gamma(t_n)), \mathcal{L}(\gamma_2(t_n))) \ge \varepsilon_0 \tag{71}$$

for any $n \in \mathbb{N}$. Using the same arguments as in the proof of Theorem 7.8 we obtain a contradiction which proves our statement. To finish the proof it is sufficient to apply Theorems 4.9 and 4.11 (see also Remark 4.15). □

Corollary 7.14. *Under the conditions of Theorem 7.13 if* (A, f, g) *is* τ*-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(A, f, g)$ *is a minimal set), then equation (53) has a unique solution* p *defined on* $\mathbb{R}$ *which is* τ*-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(A, f, g)$ *is a minimal set) in distribution.*

Proof. This statement follows from Theorem 7.13 and Corollary 4.13 (see also Corollary 4.10). □

Theorem 7.15. *Under the condition of Theorem 7.13 if we replace condition (S) (see item (i).) by asymptotic stability of equation of every equation (56), then equation (53) has a unique solution* p *defined on* $\mathbb{R}$ *which is strongly compatible in distribution and* $\lim\limits_{t \to +\infty} \beta(\mathcal{L}(p(t)), \mathcal{L}(\varphi(t, x, (A, f, g))) = 0$.

Proof. This statement can be proved using the same arguments as in the proof of Theorem 7.13 but instead of Theorem 4.3 and Corollary 4.13 it is necessary to apply Theorem 4.11 and Corollary 4.13. □

Corollary 7.16. *Under the conditions of Theorem 7.15 if* (A, f, g) *is* τ*-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(A, f, g)$ *is a minimal set), then every solution of equation (53) is asymptotically* τ*-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable and* $H(A, f, g)$ *is a minimal set) in distribution.*

Proof. This statement follows from Theorem 7.15 and Corollary 4.13. □

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