LEVITAN/BOHR ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF THE SCALAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this paper is to prove the existence of Levitan/Bohr almost periodic, almost automorphic, recurrent and Poisson stable solutions of the scalar differential equation

$$x' = f(\sigma(t, y), x), \quad (y \in Y) \tag{1}$$

where Y is a complete metric space and $(Y, \mathbb{R}_+, \sigma)$ is a one-sided dynamical system (also called a driving system). The existence of at least one quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent, Levitan almost periodic, almost recurrent, Poisson stable) solution of (1) is proved under the condition that (1) admits at least one bounded on the positively semi-axis and uniformly Lyapunov stable solution.

1. INTRODUCTION

Let X and Y be two complete metric spaces and $C(X, Y)$ be the space of all continuous functions $F : X \mapsto Y$ equipped with the compact-open topology. The aim of this paper is to analyze the existence of Levitan/Bohr almost periodic, almost automorphic, recurrent and Poisson stable solutions of the scalar differential equation

$$x' = f(\sigma(t, y), x), \quad (y \in Y)$$

where Y is a complete metric space, $(Y, \mathbb{T}, \sigma)$ is a (driving) dynamical system, $F \in C(Y \times \mathbb{R}, \mathbb{R})$, $\mathbb{R} := (-\infty, +\infty)$ (respectively, $\mathbb{R}_+ := [0, +\infty)$) and $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$.

The existence of Bohr almost periodic solutions of equation

$$x' = f(t, x) \tag{2}$$

with Bohr almost periodic right hand-side f in t, uniformly with respect to (shortly w.r.t.) x on every compact subset in $\mathbb{R}$ (see Example and definition therein) was studied by many authors.

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By B. P. Demidovich [11] was proved that if the function $g \in C(\mathbb{R}, \mathbb{R})$ is almost periodic and its primitive

$$G(t) := \int_0^t g(s)ds$$

is bounded on $\mathbb{R}$ and the function $f \in C(\mathbb{R}, \mathbb{R})$ is monotone and continuously differentiable, then every bounded on $\mathbb{R}$ solution of equation

$$x' = f(x) + g(t)$$

is also almost periodic. He also noted that this statement remains true without of the boundedness of the function G, if $f'(x) \geq k > 0$ (or $f'(x) \leq k < 0$) for all $x \in \mathbb{R}$.

Using the ideas of Demidovich, by N. Gheorghiu [15] was generalized the last result for differential equation (2). Namely he proved that, if $\varphi \in C(\mathbb{R}, \mathbb{R})$ is a solution of equation (2) and the following conditions are fulfilled:

(i) $|\varphi(t)| \leq m$ for all $t \in \mathbb{R}$;
(ii) the function $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is almost periodic in t uniformly w.r.t. $x \in [-m.m]$;
(iii) the function f is continuously differentiable in x and $f'_x(t, x) \geq k > 0$ (or $f'_x(t, x) \leq k < 0$) for all $(t, x) \in \mathbb{R} \times [-m, m]$.

Then each solution $\psi : \mathbb{R} \mapsto [-M, M]$ of equation (2) is almost periodic.

Remark 1.1. *Under the conditions above* (i)-(iii) *there exists a unique almost periodic solution* $\psi : \mathbb{R} \mapsto [-m, m]$ *of equation (2).*

Z. Opial [18] generalized results of B. P. Demidovich and N. Gheorghiu for differential equation (2) if the second hand right side is only monotone in the sense large w.r.t. spacial variable $x \in \mathbb{R}$.

By B. A. Shcherbakov [26] was studied the problem of Poisson stability (in particular, periodic, Bohr almost periodic, recurrent in the sense of Birkhoff, almost recurrent in the sense of Bebutov, Levitan almost periodic) of solutions for equation (2) with Poison stable in time $t \in \mathbb{R}$ (uniformly w.r.t. x on every compact subset from $\mathbb{R}$) right hand side f. He generalized Z. Opial's result for Poisson stable differential equations (2).

Let E be a Banach space with the norm $|\cdot|$. Consider differential equation (2) in the Banach space E, i.e., we suppose that $f \in C(\mathbb{R} \times E, E)$. By V. V. Zhikov [31] was studied a special class of differential equations (2) in the strict convex Banach spaces (so called V-monotone systems). Recall that equation (2) is said to be V–monotone w.r.t. $x \in E$, if there exists a continuous non-negative function $V : E \times E \to \mathbb{R}_+$, which equals to zero only on the diagonal, so that the numerical function $\alpha(t) := V(x_1(t), x_2(t))$ is non-increasing w.r.t. $t \in \mathbb{R}_+$, where $x_1(t)$ and $x_2(t)$ are two arbitrary solutions of (2) defined on $\mathbb{R}_+$. V. V. Zhikov generalized for V-monotone systems in the strictly convex Banach space the result of Z. Opial.

In the work of D. Cheban [8] the results of V. V. Zhikov and B. A. Shcherbakov was generalized for equations (2) in the Banach space E with arbitrary Poisson stable (w.r.t. time t) second hand right side f.

Finally, below we note some of works, where one study the problem of almost periodicity of solutions of scalar almost periodic equation (2) without assumption of monotony of the second hand right side f w.r.t. spacial variable $x \in \mathbb{R}$.

A solution $\varphi(t, x_0, f)$ of equation (2) is called distal (in the positive direction), if

$$\inf_{t \ge 0} |\varphi(t, x_0, f) - \varphi(t, x, f)| > 0$$

for all $x \neq x_0$ (with $\varphi(\mathbb{R}, x, f) \subseteq Q := \overline{\varphi(\mathbb{R}, x_0, f)}$).

V. V. Zhikov [30] (see also [2, ChIV] and [17, ChVII]) established that equation (2) with almost periodic second right hand side f admits at least one almost periodic solution, if it admits a bounded on $\mathbb{R}$ distal solution.

By R. Sacker and G. Sell [19, 21] (see also [14, ChXI]) was proved that equation (2) with almost periodic coefficients has at least one almost periodic solution, if it admits a bounded on $\mathbb{R}_+$ uniformly Lyapunov stable solution.

Remark 1.2. *1. If the second right hand side f of equation (2) is regular, almost periodic and $\varphi(t, x, f)$ is a bounded on $\mathbb{R}_+$ uniform Lyapunov stable solution of (2), then it admits a bounded on $\mathbb{R}$ uniformly stable solution* [22] *(see also* [14, ChXI] *and* [23, Ch]*).*

2. Under the conditions of item 1, if $\varphi(t, x, f)$ is a bounded on $\mathbb{R}$ and uniformly Lyapunov stable solution of equation (2), then it is distal [30] *(see also* [17, ChVII]*). Thus the result of R. Sacker and G. Sell follows from the more early result of V. V. Zhikov.*

In general case the proof of the existence of an almost periodic solution (under the assumption that a bounded solution exists on R) turns out to be difficult. For example, the difficulty consists in the fact that equation (2) might have an infinite number of bounded solutions on R (for instance, all solutions might be bounded on R) and it is not clear how should we pick an almost periodic solution out of this set of bounded solutions.

The aim of this paper is studying the problem of existence of Levitan/Bohr almost periodic (respectively, almost automorphic, recurrent and Poisson stable) solutions of the scalar differential equation (2), when the second right hand side is not monotone with respect to spacial variable. The existence at least one quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent, Levitan almost periodic, almost recurrent, Poisson stable) solution of (2) is proved under the condition that (2) admits at least one bounded on the positively semi-axis solution $\varphi(t, x_0, f)$ and one of the following two conditions holds:

(i) the solution $\varphi(t, x_0, f)$ is uniformly stable;
(ii) the solutions of equation (2) with the values from $Q := \overline{\varphi(\mathbb{R}, x_0, f)}$ are distal.

The paper is organized as follows.

In Section 2 we collected some notions and facts from the theory of dynamical systems which we use in this paper: Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable motions and functions, cocycles, skew-product dynamical systems, non-autonomous dynamical systems.

Section 3 is dedicated to the study the problem of existence a common fixed point for some semigroup of nonlinear transformations. The main result of this section is Theorem 3.12, where we establish the existence at least one fixed point for a semi-group consisting from strict monotone increasing maps acting on a one-dimensional compact subset. This result we apply in Section 4 for one-dimensional non-autonomous dynamical systems in the studying their Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable motions.

In Section 4 we study the comparable in the sense of B. A.Shcherbacov motions of dynamical systems by character of their recurrence. Comparability the motions by character of recurrence plays a very important role in the study the problem of existence of Bohr/Levitan almost periodic (respectively, quasi-periodic, almost automorphic, almost recurrent, recurrent and Poisson stable) solutions of the different types of evolution equations with Poisson stable coefficients. The main results in this sections are Theorems 4.16 and 4.20 which contain simple conditions of the existence of comparable (respectively, uniformly comparable) motions for one-dimensional non-autonomous dynamical systems.

Section 5 is dedicated to the study the problem of existence of Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable solutions of scalar non-autonomous differential equation (2). Taking into consideration that equation (2) under some appropriate conditions generates some non-autonomous one-dimensional dynamical system we are able to apply our general results from Sections 3 and 4 to the study the problem of existence of different classes of Poisson stable solutions (as Bohr/Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent in the sense of Birkghoff) of differential equation (2). In this way we obtain a series of new results (some of them coincides with the well known results).

2. Bohr/Levitan Almost Periodic and Almost Automorphic Motions of Dynamical Systems

In this section we recall some notions, facts and constructions from the theory of dynamical systems. In order to keep our paper self-contained as much as possible, we prefer to include the necessary results.

Let $(X, \mathbb{T}, \pi)$ be *a dynamical system* on the complete metric space X, i.e., let $\pi : \mathbb{T} \times X \to X$ be a continuous function such that $\pi(0, x) = x$ for all $x \in X$, and $\pi(t_1 + t_2, x) = \pi(t_2, \pi(t_1, x))$, for all $x \in X$, and $t_1, t_2 \in \mathbb{T}$.

2.1. **Recurrent, Bohr Almost Periodic and Almost Automorphic Motions.**

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x, if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{\pi(t,x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n, x)\} \to x \text{ and } \{t_n\} \to \infty\}$ and $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n, x)\} \text{ is convergent and } \{t_n\} \to \infty\}$.

A point $x \in X$ is called *Poisson stable in the positive direction* if there exists a sequence $\{t_n\} \in \mathfrak{N}_x$ such that $t_n \to +\infty$ as $n \to \infty$.

A dynamical system $(X, \mathbb{T}, \pi)$ is said to be

(i) *transitive*, if there exists a point $x_0 \in X$ such that $H(x_0) = X$, where $H(x_0) := \overline{\{\pi(t, x_0) : \ t \in \mathbb{T}\}}$;
(ii) *pseudo recurrent* if X is compact, transitive and every point $x \in X$ is Poisson stable.

A point $x \in X$ is called [25, 27] *pseudo recurrent* if the dynamical system $(H(x), \mathbb{T}, \pi)$ is pseudo recurrent.

Remark 2.1. *Every recurrent point is pseudo recurrent, but there exist pseudo recurrent points which are not recurrent* [25, 27].

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$ with the frequency $\nu = (\nu_1, \nu_2, \dots, \nu_m) \in \mathbb{R}^m$, i.e., $\sigma(t, v) := (v_1 + \nu_1 t(mod\ 2\pi), v_2 + \nu_2 t(mod\ 2\pi), \dots, v_m + \nu_m t(mod\ 2\pi))$ for all $t \in \mathbb{T}$ and $v = (v_1, v_2, \dots, v_m) \in \mathcal{T}^m$, where the numbers $\nu_1, \nu_2, \dots, \nu_m$ are rational independent.

A point $x \in X$ is called *quasi-periodic* with the frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m) \in \mathbb{R}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ such that $\pi(t, x) := \Phi(\sigma(t, v))$ for all $t \in \mathbb{T}$, where $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$ with the frequency $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ and $v \in \mathcal{T}^m$ such that $\Phi(v) = x$.

A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called *Levitan almost periodic* [17], if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.2. *Let $x_i \in X_i$ $(i = 1, 2, \dots, m)$ be a Levitan almost periodic point of the dynamical system $(X_i, \mathbb{T}, \pi_i)$. Then the point $x := (x_1, x_2, \dots, x_m)) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic in the product dynamical system $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t, x) := (\pi_1(t, x_1), \pi_2(t, x_2), \dots, \pi_m(t, x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1, x_2, \dots, x_m) \in X$.*

A point $x \in X$ is called *stable in the sense of Lagrange* (st.L) (respectively, *stable in the sense of Lagrange in the positive direction* ($st.\mathrm{L}^+$)), if its trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ (respectively, its positive semi-trajectory $\{\pi(t, x) \ : \ t \in \mathbb{T}_+\}$) is relatively compact, where $\mathbb{T}_+ := \{t \in \mathbb{T} : \ t \geq 0\}$.

A point $x \in X$ is called *almost automorphic* [17, 28] in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

A motion $\pi(\cdot, x)$ of dynamical system $(X, \mathbb{T}, \pi)$ is called stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, almost recurrent, Poisson stable) if the point $x \in X$ is so.

2.2. **Shift Dynamical Systems, Levitan/Bohr Almost Periodic and Almost Automorphic Functions.** Below we recall a general method of construction of dynamical systems on spaces of continuous functions. In this way, we will obtain many well-known dynamical systems on some functional spaces (see, for example, [2, 23, 25]).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on the complete metric space X, Y be a complete pseudo metric space, and $\mathcal{P}$ be a family of pseudo metrics on Y. We denote by $C(X, Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d_K^p\}$ $(p \in \mathcal{P},\ K \in \mathcal{C}(X))$, where

$$d_K^p(f, g) := \sup_{x \in K} p(f(x), g(x))$$

and $\mathcal{C}(X)$ denotes the family of all compact subsets of X. For all $\tau \in \mathbb{T}$ we define the mapping $\sigma_\tau : C(X, Y) \to C(X, Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau, x)),\ x \in X$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = Id_{C(X,Y)}$;
b. $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1 + \tau_2}$, for all $\tau_1, \tau_2 \in \mathbb{T}$;
c. σ_τ is continuous for all $\tau \in \mathbb{T}$.

Lemma 2.3. [7] *The mapping* $\sigma : \mathbb{T} \times C(X, Y) \to C(X, Y)$, *defined by the equality* $\sigma(\tau, f) := \sigma_\tau f \quad (f \in C(X, Y),\ \tau \in \mathbb{T})$, *is continuous.*

Corollary 2.4. *The triple* $(C(X, Y), \mathbb{T}, \sigma)$ *is a dynamical system on* $C(X, Y)$.

Consider now two examples of dynamical systems of the form $(C(X, Y), \mathbb{T}, \sigma)$, which are useful in the applications.

Example 2.5. Let $X = \mathbb{T}$, and denote by $(X, \mathbb{T}, \pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t, x) := x + t$. The dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ is called *Bebutov's dynamical system* [2, 23, 25] (a dynamical system of translations, or shifts dynamical system).

It is said that the function $\varphi \in C(\mathbb{T}, Y)$ *possesses a property* (A), if the motion $\sigma(\cdot, \varphi) : \mathbb{T} \to C(\mathbb{T}, Y)$, generated by this function, possesses this property in the Bebutov dynamical system $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$. As property (A) we can take periodicity, quasi-periodicity, Bohr/Levitan almost periodicity, almost automorphy, recurrence, pseudo recurrence, Poisson stability etc.

Example 2.6. Let $X := \mathbb{T} \times W$, where W is a metric space, and let $(X, \mathbb{T}, \pi)$ denote a dynamical system on X defined in the following way: $\pi(t, (s, w)) := (s+t, w)$. Using the general method proposed above, we can define on $C(\mathbb{T} \times W, Y)$ a dynamical system of translations $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

The function $f \in C(\mathbb{T} \times W, Y)$ is called *Bohr/Levitan almost periodic (quasi-periodic, recurrent, almost automorphic, etc) in $t \in \mathbb{T}$, uniformly w.r.t. w on every compact subset from W*, if the motion $\sigma(\cdot, f)$ is Bohr/Levitan almost periodic (quasi-periodic, recurrent, almost automorphic, etc.) in the dynamical system $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

Remark 2.7. *Notice the following well-known facts.*

1. *Every almost automorphic point is Levitan almost periodic.*

2. *A Levitan almost periodic point is almost automorphic if and only if it is stable in the sense of Lagrange.*

3. *Let*
$$\varphi(t) := \frac{1}{2 + \sin t + \sin \sqrt{2}t}$$
for all $t \in \mathbb{R}$, then the point $\varphi \in C(\mathbb{R}, \mathbb{R})$ is Levitan almost periodic with respect to Bebutov's dynamical system $(C(\mathbb{R}, \mathbb{R}), \mathbb{R}, \sigma)$, but it is not almost automorphic.

2.3. **Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems.** Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{R}$ ($\mathbb{R}_+ \subseteq \mathbb{T}_1$).

A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$, where h is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ (i.e., h is continuous and $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$ and $x \in X$), is called a *non-autonomous dynamical system.*

Let $(Y, \mathbb{T}_2, \sigma)$ be a dynamical system on Y, W be a complete metric space and φ be a continuous mapping from $\mathbb{T}_1 \times W \times Y$ into W, possessing the following properties:

a. $\varphi(0, u, y) = u$ $(u \in W, y \in Y)$;
b. $\varphi(t + \tau, u, y) = \varphi(\tau, \varphi(t, u, y), \sigma(t, y))$ $(t, \tau \in \mathbb{T}_1,\ u \in W, y \in Y)$.

Then the triplet $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ (or shortly φ) is called [23] *a cocycle* over $(Y, \mathbb{T}_2, \sigma)$ with the fiber W.

Let $X := W \times Y$ and let us define a mapping $\pi : X \times \mathbb{T}_1 \to X$ as follows: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e., $\pi = (\varphi, \sigma)$). Then, it is easy to see that $(X, \mathbb{T}_1, \pi)$ is a dynamical system on X, which is called a *skew-product dynamical system* [23] and $h = pr_2 : X \to Y$ is a homomorphism from $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ and, hence, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ over the dynamical system $(Y, \mathbb{T}_2, \sigma)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ $(X := W \times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W, \varphi, (Y, \mathbb{T}_2, \sigma) \rangle$ on $(Y, \mathbb{T}_2, \sigma)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions, every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system). Below we give some examples of this type using in this paper.

Example 2.8. Let $(Y,\mathbb{T},\sigma)$ be a dynamical system on the metric space Y (driving system). We consider the equation

$$u' = F(\sigma(y,t),u) \quad (y \in Y), \tag{3}$$

where $F \in C(Y \times \mathbb{R}^n,\mathbb{R}^n)$. Suppose that for equation (3) the conditions for the existence, uniqueness and extendability of solutions to $\mathbb{R}_+$ are fulfilled. The non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ (respectively, the cocycle $\langle \mathbb{R}^n,\varphi,(Y,\mathbb{T},\sigma)\rangle$), where $X := \mathbb{R}^n \times Y$, $\pi := (\varphi,\sigma)$, $\varphi(\cdot,x,y)$ is the solution of (3) and $h := pr_2 : X \to Y$, is generated by equation (3).

A solution $\varphi(t,u,y)$ of equation (3) is called [25, 27] *compatible (respectively, uniformly compatible) by the character of recurrence* if $\mathfrak{N}_y \subseteq \mathfrak{N}_u$ (respectively, $\mathfrak{M}_y \subseteq \mathfrak{M}_u$), where $\mathfrak{N}_u$ (respectively, $\mathfrak{M}_u$) is the set of all sequences $\{t_n\} \subset \mathbb{R}$ such that $\{\varphi(t+t_n,u,y\}$ converges to $\varphi(t,u,y)$ (respectively, $\{\varphi(t+t_n,u,y\}$ converges) in the space $C(\mathbb{T},\mathbb{R}^n)$.

Remark 2.9. *The sequence $\{\varphi(t+t_n,u,y)\}$ converges to the function ψ in the space $C(\mathbb{T},\mathbb{R}^n)$ if and only if $\{\varphi(t_n,u,y)\}$ converges to $\psi(0)$.*

Example 2.10. We consider the equation

$$u' = f(t,u), \tag{4}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^n,\mathbb{R}^n)$. Along with equation (4) we consider the family of equations

$$u' = g(t,u), \tag{5}$$

where $g \in H(f) := \overline{\{f_\tau :\ \tau \in \mathbb{R}\}}$, by bar is denoted the closure in $C(\mathbb{R} \times \mathbb{R}^n,\mathbb{R}^n)$ and f_τ is the τ-shift of f w.r.t. time, i.e., $f_\tau(t,u) := f(t+\tau,u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^n$. Suppose that the function f is regular [23], i.e., for all $g \in H(f)$ and $u \in \mathbb{R}^n$ there exists a unique solution $\varphi(t,u,g)$ of equation (5) defined on $\mathbb{R}_+$. Denote by $Y = H(f)$ and $(Y,\mathbb{R},\sigma)$ a shift dynamical system on Y induced by the Bebutov dynamical system $(C(\mathbb{R} \times \mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$. Now the family of equations (5) can be written as (3) if we take the mapping $F \in C(Y \times \mathbb{R}^n,\mathbb{R}^n)$ defined by $F(g,u) := g(0,u)$, for all $g \in H(f)$ and $u \in \mathbb{R}^n$.

Theorem 2.11. [25, 27] *The following statements hold:*

1. Let $y \in Y$ be a stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable) point. If $\varphi(t,u,y)$ is a compatible solution of equation (3), then so is $\varphi(t,u,y)$.

2. Let $y \in Y$ be a stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent) point. If $\varphi(t,u,y)$ is a uniformly compatible solution of equation (3), then so is $\varphi(t,u,y)$.

3. Some criterion of the existence of fixed point for a semigroup of transformations

In this section we will prove a general theorem of existence of common fixed point for a semigroup of transformations of a compact subset K from $\mathbb{R}$. This fact we will use in Section 4 to prove the existence of comparable (respectively, uniform comparable) motions by character of their recurrence for one dimensional non-autonomous dynamical systems.

Denote by 2^X the family of all compact subset of X equipped with the Hausdorff metric. Let $F : X \mapsto 2^X$ be a set-valued map, that is, $\emptyset \neq F(x) \in 2^X$ for all $x \in X$.

Definition 3.1. *The mapping $F : X \mapsto 2^X$ is said to be compact if the set $F(M)$ is compact for all $M \in 2^X$, where $F(M) := \bigcup\{F(x) : x \in M\}$.*

It is well known (see, for example, [10, ChI]) that, if the map $F : X \mapsto 2^X$ is upper semi-continuous, then it is compact.

Definition 3.2. *A subset $M \subseteq X$is said to be:*

- *F-invariant, if $F(M) \subseteq M$;*
- *F-minimal if M is non-empty, F-invariant, closed and it does not contain an own closed F-invariant subset.*

Lemma 3.3. *Suppose that $F : X \mapsto 2^X$, $K \in 2^X$ and the following conditions are fulfilled:*

(i) *the set K is F-invariant;*
(ii) *the mapping F is compact;*
(iii) *$F^2(x) \subseteq F(x)$ for all $x \in K$, where $F^2(x) := F(F(x))$.*

Then there exists a nonempty, compact F-minimal subset $M \subseteq K$.

Proof. Denote by K^* the family of all nonempty F-invariant compact subsets $A \subseteq K$. Note that $K^* \neq \emptyset$ because $K \in K^*$. It is clear that the family K^* partially ordered with respect to the inclusion $\subseteq$. Namely: $A_1 \leq A_2$ if and only if $A_1 \subseteq A_2$ for all $A_1, A_2 \in K^*$. If $\mathcal{K} := \{K_\lambda : \ \lambda \in \Lambda\} \subseteq K^*$ is a linear ordered subfamily of K^*, then the intersection B of subsets of the family $\mathcal{K}$ is nonempty, since the set K is compact. Note that the set B is F-invariant because

$$F(B) = F(\bigcap_{\lambda\in\Lambda} K_\lambda) \subseteq \bigcap_{\lambda\in\Lambda} F(K_\lambda) \subseteq \bigcap_{\lambda\in\Lambda} K_\lambda = B.$$

Thus $B \in K^*$. By Lemma of Zorn the family K^* contains at least one minimal element M. It is clear that M is a F-minimal set. Lemma is proved. □

Lemma 3.4. *Under the conditions of Lemma 3.3 the set $M \in 2^X$ is F-minimal if and only if $F(x) = M$ for all $x \in M$.*

Proof. Let $M \in 2^X$ be F-minimal and x be an arbitrary element from M. By conditions of Lemma we have $F(F(x)) \subseteq F(x)$ and, consequently, the set $F(x)$ is an F-invariant subset. Since the set M is F-minimal, then $F(x) = M$.

Let now $M' \subseteq M$ be an arbitrary F-invariant, closed subset of M and $x_0 \in M'$, then $M = F(x_0) \subseteq M'$ and, consequently, $M' = M$. □

If $K \in 2^X$, then K^K denotes the collection of all maps from K to itself, provided with the product topology, or, what is the same thing, the topology of pointwise convergence. By Tychonoff's theorem, K^K is compact. K^K has a semi-group structure defined by the composition of maps.

Let $\mathcal{E}$ be a semi-group. A right ideal in $\mathcal{E}$ is a non-empty subset I such that $\mathcal{E}I \subset I$, where $\mathcal{E}I := \{\eta \circ \xi \ : \xi \in I,\ \eta \in \mathcal{E}\}$ and $\eta \circ \xi$ is a composition of η and ξ, i.e., $(\eta \circ \xi)(x) := \eta(\xi(x))$ for all $x \in K$.

A minimal right ideal is one which does not properly contain a right ideal.

Remark 3.5. *1. Every compact semigroup admits at least one minimal right ideal* [1, 2].

2. Every compact semigroup $\mathcal{E}$ contains at least one idempotent element [1, 2]*, i. e., an element u with $u^2 = u$.*

Lemma 3.6. *Suppose that $K \in 2^X$, $\mathcal{E} \subseteq K^K$ be a compact sub-semigroup, then the compact right ideal $I \subseteq \mathcal{E}$ is minimal if and only if $\mathcal{E}\xi = I$ for each $\xi \in I$.*

Proof. Note that the compact semi-group $I \subseteq \mathcal{E}$ is a minimal right ideal if and only if it is an F-minimal subset of $\mathcal{E}$, where $F(\xi) := \mathcal{E}\xi$ for all $\xi \in \mathcal{E}$. Let $\xi \in I$, then $F(\xi) \in 2^{\mathcal{E}}$ because the left multiplication in $\mathcal{E} \subseteq K^K$ (i.e., the mapping $\xi \mapsto \xi \circ \eta$ for every η fixed) is continuous and $\mathcal{E}$ is compact. Note that

$$F^2(\xi) = F(F(\xi)) = \mathcal{E}F(\xi) = \mathcal{E}\mathcal{E}\xi \subseteq \mathcal{E}\xi = F(\xi)$$

for all $\xi \in \mathcal{E}$. Now to finish the proof it is sufficient to apply Lemma 3.4. □

Definition 3.7. *Let $K \in 2^X$ and $\mathcal{E} \subseteq K^K$ be a compact sub-semigroup. A subset $A \subseteq K$ is said to be $\mathcal{E}$-invariant, if $\mathcal{E}A \subseteq A$, where $\mathcal{E}A := \bigcup\{\xi(A) :\ \xi \in \mathcal{E}\}$.*

Lemma 3.8. *If $A \subseteq K$ is a compact and $\mathcal{E}$-invariant, then A contains a nonempty compact $\mathcal{E}$-minimal subset $M \subseteq A$.*

Proof. Consider the mapping $\Phi : A \mapsto 2^A$ defined by the equality $\Phi(x) = \mathcal{E}x := \{\xi(x) :\ \xi \in \mathcal{E}\}$. Note that

$$\Phi^2(x) = \Phi(\Phi(x)) = \Phi(\mathcal{E}x) = \bigcup\{\Phi(\xi(x)) :\ \xi \in \mathcal{E}\} =$$

$$\bigcup\{\mathcal{E}\xi(x) :\ \xi \in \mathcal{E}\} = \mathcal{E}\mathcal{E}x \subseteq \mathcal{E}x = \Phi(x)$$

for all $x \in A$. By Lemma 3.3 in the set A there exists at least one nonempty compact Φ-minimal subset $A_0 \subseteq A$. It is clear that the set A_0 is $\mathcal{E}$-minimal. □

Lemma 3.9. *Under the conditions of Lemma 3.3 the following conditions are equivalent:*

(i) *the compact set $M \subseteq K$ is $\mathcal{E}$ minimal;*
(ii) *$\mathcal{E}(x) = M$for all $x \in M$.*

Proof. Let $M \subseteq K$ be a compact $\mathcal{E}$-minimal set and $x \in M$ be an arbitrary point. Then reasoning as well as in the proof of Lemma 3.8 we can show that the set $\mathcal{E}(x)$ a nonempty, compact and $\mathcal{E}$-invariant subset of K. Since the set M is $\mathcal{E}$-minimal, then $\mathcal{E}(x) = M$.

Let now $\mathcal{E}(x) = M$ for all $x \in M$. If we suppose that the compact set M is not $\mathcal{E}$-minimal, then there exists a nonempty, compact proper subset $M' \subset M$ $(M' \neq M)$ which is $\mathcal{F}$-invariant, where $\mathcal{F}(x) := \mathcal{E}(x)$ for all $x \in M$. Let now $x' \in M'$, then $\mathcal{E}(x')$ is a nonempty, compact and $\mathcal{F}$-invariant subset of M', i.e.,

$$M = \mathcal{E}(x') \subseteq M' \subseteq M$$

and, consequently $M = M'$. The obtained contradiction proves our statement. □

Below we will establish the relation between the $\mathcal{E}$-minimal subsets in K and the minimal right ideals in $\mathcal{E}$.

Lemma 3.10. *Under the conditions of Lemma 3.3 the following statements hold:*

(i) *If $I \subseteq \mathcal{E}$ is a compact minimal right ideal and $x \in K$, then the set $I(x)$ is $\mathcal{E}$-minimal;*
(ii) *If $M \subseteq K$ is a compact $\mathcal{E}$-minimal set, $x \in M$ and $I \subseteq \mathcal{E}$ is an arbitrary compact minimal right ideal, then $M = I(x)$.*

Proof. Let $x' \in I(x)$, then there exists $\xi \in I$ such that $x' = \xi(x)$. Note that by Lemma 3.6 we have $\mathcal{E}\xi = I$ for all $\xi \in I$ and, consequently, $\mathcal{E}x' = \mathcal{E}\xi(x) = I(x)$ for all $x' \in M$. Thus we have $\mathcal{E}(x') = I(x)$ for all $x' \in I(x)$ and by Lemma 3.9 we conclude that the set $I(x)$ is $\mathcal{E}$-minimal.

Let now $M \subseteq K$ be a compact $\mathcal{E}$-minimal set, $x \in M$ and $I \subseteq \mathcal{E}$ be an arbitrary compact minimal right ideal. Consider the nonempty compact set $M' = I(x) \subseteq \mathcal{E}(x) = M$. Note that $\mathcal{E}(M') = \mathcal{E}I(x) \subseteq I(x)$ because $\mathcal{E}I \subseteq I$. Thus $M' = I(x)$ is a nonempty, compact and $\mathcal{E}$-invariant subset of M. By $\mathcal{E}$-minimality of M we obtain $I(x) = M$. □

Corollary 3.11. *Let $K \in 2^X$ be a compact subset, $\mathcal{E} \subseteq K^K$ be a sub-semigroup, $I \subset \mathcal{E}$ be a compact minimal right ideal of $\mathcal{E}$ and $M \subseteq K$ be a nonempty compact $\mathcal{E}$-minimal subset, then $M = I(x) = \mathcal{E}(x)$ for all $x \in M$.*

Theorem 3.12. *Let $K \in 2^X$ and $\mathcal{E}$ be a nonempty compact sub-semigroup of K^K. If the compact set K is one-dimensional (i.e., $K \subset \mathbb{R}$ or K is homeomorphic to a compact subset from $\mathbb{R}$) and every mapping $\xi \in \mathcal{E}$ is strictly monotone increasing, then there exists at least one fixed point $x_0 \in K$ of sub-semigroup $\mathcal{E}$, i.e., $\xi(x_0) = x_0$ for all $\xi \in \mathcal{E}$.*

Proof. By Lemma 3.8 the compact set K contains at least one nonempty compact $\mathcal{E}$-minimal subset M. From Corollary 3.11 we have

$$M = \mathcal{E}(x). \tag{6}$$

for all $x \in M$. We will show that the set M consists of a single point x_0, i.e., $M = \{x_0\}$. If we suppose that it is not true, then there exist at least two different points $x_1, x_2 \in M$. Suppose, for example, that $x_1 < x_2$. Denote by $\alpha := \inf M$ and $\beta := \sup M$. Since M is a compact set, then $\alpha, \beta \in M$ and, consequently,

$$\alpha \le x_1 < x_2 \le \beta.$$

From equality (6) we have $M = \mathcal{E}(\beta)$ and, consequently, there exists a mapping $\eta \in \mathcal{E}$ such that

$$\eta(\beta) = \alpha. \tag{7}$$

According to the conditions of Theorem the mapping $\eta : M \mapsto M$ is strictly monotone increasing, i.e.,

$$\alpha \le \eta(\alpha) < \eta(\beta) \le \beta. \tag{8}$$

From (7) and (8) we obtain $\alpha \le \eta(\alpha) < \eta(\beta) = \alpha$. The obtained contradiction proves our statement. Thus $M = \{x_0\}$ and, consequently, $x_0 = \xi(x_0)$ for all $\xi \in \mathcal{E}$. □

Remark 3.13. *1. Note that we proved in fact that under the conditions of Theorem 3.12 every $\mathcal{E}$-minimal set M consists of a single point.*

2. It easy to see that Theorem 3.12 remains true if every mappings $\xi \in \mathcal{E}$ is strictly decreasing.

3. It easy to see (reasoning as in the proof of Theorem 3.12) that Theorem 3.12 remains true also in the case when K is a subset from a partially ordered Banach space, if the following conditions hold:

- (i) *K is a nonempty compact subset;*
- (ii) *there are $\alpha, \beta \in K$ such that $\alpha \le x \le \beta$ for all $x \in K$;*
- (iii) *every mapping $\xi \in \mathcal{E}$ is strict monotone increasing.*

4. Comparable and uniform comparable motions by character of recurrence

In this section we study the comparable in the sense of B. A.Shcherbacov motions of dynamical systems by character of their recurrence. Comparability the motions by character of their recurrence plays a very important role in the study the problem of existence of Bohr/Levitan almost periodic (respectively, quasi-periodic, almost automorphic, almost recurrent, recurrent and Poisson stable) solutions of the different types of evolution equations with Poisson stable coefficients.

4.1. **Comparability and uniform comparability of motions in the since of Shcherbakov.** Let (X, h, Y) be a fiber space, i.e., X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X onto Y.

A subset $M \subseteq X$ is said to be conditionally relatively compact [6, 7], if the pre-image $h^{-1}(Y') \bigcap M$ of every relatively compact subset $Y' \subseteq Y$ is a relatively compact subset of X, in particularly $M_y := h^{-1}(y) \bigcap M$ is relatively compact for every y. The set M is called conditionally compact if it is closed and conditionally relatively compact.

Lemma 4.1. [4] *Let $\langle W,\varphi,Y,\mathbb{T},\sigma)\rangle$ be a cocycle and $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system associated by cocycle φ. Suppose that $x_0 := (u_0,y_0)\in X := W\times Y$ and the set $Q_{(u_0,y_0)} := \overline{\{\varphi(t,u_0,y_0)\mid t\in\mathbb{R}\}}$ (respectively, $Q^+_{(u_0,y_0)} := \overline{\{\varphi(t,u_0,y_0)\mid t\in\mathbb{R}_+\}}$ is compact.*

Then the set $H(x_0) := \overline{\{\pi(t,x_0)\mid t\in\mathbb{R}\}}$ (respectively, $\overline{\{\pi(t,x_0)\mid t\in\mathbb{R}_+\}} := H^+(x_0)$) is conditionally compact.

Let $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system, $M\subseteq X$ be a nonempty, closed and positively invariant subset, and $y\in Y$ be a positively Poisson stable point. Denote by

$$\mathcal{E}^+_y := \{\xi|\quad \exists\{t_n\}\in\mathfrak{N}^{+\infty}_y \quad\text{such that}\quad \pi^{t_n}|_{M_y}\to\xi\},$$

where $\pi^t := \pi(t,\cdot)$, $X_y := \{x\in X|\quad h(x)=y\}$, $M_y := M\bigcap X_y$, $\to$ means the pointwise convergence and $\mathfrak{N}^{+\infty}_y := \{\{t_n\}\in\mathfrak{N}_y$ such that $t_n\to+\infty$ as $n\to\infty\}$.

Lemma 4.2. [6, 7] *Let $y\in Y$ be a positively Poisson stable point, $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system and M be a conditionally compact space, then $\mathcal{E}^+_y$ is a nonempty compact sub-semigroup of the semigroup $M_y^{M_y}$ (w.r.t. composition of mappings).*

Theorem 4.3. [10, ChVI] *Let X be a conditionally compact metric space and $\langle(X,\mathbb{R}_+,\pi),\ (Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $y\in Y$ is positively Poisson stable;*
(ii) *$\lim\limits_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$ for all $x_1,x_2\in X_y := h^{-1}(y)=\{x\in X\ :\ h(x)=y\}$.*

Then there exists a unique point $x_y\in X_y$ such that $\xi(x_y)=x_y$ for all $\xi\in\mathcal{E}^+_y$.

Corollary 4.4. *Let $\langle(X,\mathbb{R}_+,\pi),\ (Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system and $x_0\in X$. Suppose that the following conditions are fulfilled:*

(i) *the set $H^+(x_0) := \overline{\{\pi(t,x_0)\mid t\in\mathbb{R}_+\}}$ is conditionally compact;*
(ii) *the point $y := h(x_0)\in Y$ is positively Poisson stable;*
(iii) *$\lim\limits_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$ for all $x_1,x_2\in H^+(x_0)\cap X_y$, where $X_y = h^{-1}(y) := \{x\in X\ :\ h(x)=y\}$.*

Then there exists a unique point $x_y\in H^+(x_0)\cap X_y$ such that $\xi(x_y)=x_y$ for all $\xi\in\mathcal{E}^+_y$.

Let $(X,\mathbb{T}_1,\pi))$ and $(Y,\mathbb{T}_2,\sigma)$ be two dynamical systems on the metric spaces X and Y respectively. A point $x\in X$ is called [25]–[27] comparable by the character of recurrence with $y\in Y$ if $\mathfrak{N}_y\subseteq\mathfrak{N}_x$.

Remark 4.5. *If a point $x\in X$ is comparable by the character of recurrence with $y\in Y$ and y is stationary (respectively, τ-periodic, almost recurrent, Levitan almost periodic, Poisson stable), then the point x is also so* [27].

Corollary 4.6. *Let X be a conditionally compact metric space and $\langle (X,\mathbb{T},\pi), (Y,\mathbb{T},\sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $y\in Y$ is positively Poisson stable;*
(ii) $\lim\limits_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$ *for all* $x_1,x_2\in X_y:=h^{-1}(y)=\{x\in X : h(x)=y\}$.

Then there exists a unique point $x_y\in X_y$ which is comparable by the character of recurrence with $y\in Y$ such that

$$\lim_{t\to+\infty}\rho(\pi(t,x),\pi(t,x_y))=0 \tag{9}$$

for all $x\in X_y$.

Corollary 4.7. *Let $y\in Y$ be a stationary (respectively, τ-periodic, almost recurrent, Levitan almost periodic, Poisson stable) point. Then under the conditions of Corollary 4.6 there exists a unique stationary (respectively, τ-periodic, almost recurrent, Levitan almost periodic, Poisson stable) point $x_y\in X_y$ such that equality (9) holds for all $x\in X_y$.*

Proof. This statement directly follows from Corollary 4.6 and Remark 4.5. □

Denote by $\mathfrak{M}_y^{+\infty}:=\{\{t_n\}\in\mathfrak{M}_y|$ such that $t_n\to+\infty$ as $n\to\infty\}$.

A point $x\in X$ is called [25]–[27] uniformly comparable by the character of recurrence with $y\in Y$ if $\mathfrak{M}_y\subseteq\mathfrak{M}_x$.

Remark 4.8. *Let $y\in Y$ be a positively Poisson stable point of dynamical system $(Y,\mathbb{T},\sigma)$ and $x\in X$ be a point of dynamical system $(X,\mathbb{T},\pi)$, then the following statements hold* [5]*:*

(i) $\mathfrak{N}_y\subseteq\mathfrak{N}_x$ *if and only if* $\mathfrak{N}_y^{+\infty}\subseteq\mathfrak{N}_x^{+\infty}$;
(ii) $\mathfrak{M}_y\subseteq\mathfrak{M}_x$ *if and only if the following two inclusions take place:*
  (a) $\mathfrak{M}_y^{+\infty}\subseteq\mathfrak{M}_x^{+\infty}$;
  (b) $\mathfrak{N}_y^{+\infty}\subseteq\mathfrak{N}_x^{+\infty}$.

In the proof of this statement one use some ideas from [9, ChI] and [27, ChII].

Remark 4.9. *1. If a point $x\in X$ is uniformly comparable by the character of recurrence with $y\in Y$ and y is stationary (respectively, τ-periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent, Poisson stable), then the point x is also so* [25]–[27].

2. Every almost automorphic (respectively, almost periodic) point is recurrent.

Theorem 4.10. [10, ChVI] *Let X be a compact metric space and $\langle (X,\mathbb{R}_+,\pi), (Y,\mathbb{T},\sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $y\in Y$ is recurrent;*
(ii) $\lim\limits_{t\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$ *for all $x_1,x_2\in X$ such that $h(x_1)=h(x_2)$.*

Then there exists a unique point $x_y \in X_y$ which is uniformly comparable by the character of recurrence with $y \in Y$ such that (9) takes place for all $x \in X_y$.

Corollary 4.11. *Let $y \in Y$ be a stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, pseudo recurrent, Poisson stable) point. Then under the conditions of Theorem 4.10 there exists a unique stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, pseudo recurrent, Poisson stable) point $x_y \in X_y$ such that (9) is fulfilled for all $x \in X_y$.*

4.2. **Comparable and uniform comparable motions by character of their recurrence for one-dimensional non-autonomous dynamical systems.** Consider a non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$. In this section everywhere we suppose that it is one-dimensional, i.e., it satisfies the following condition:

***Condition* (D1)**: For all $y \in Y$ the fiber $X_y = \mathbb{R} \times \{y\}$ or X_y is homeomorphic to $\mathbb{R} \times \{y\}$.

Let $x_i = (u_i, y) \in X_y$, then we will say that $x_1 \le x_2$, if $u_1 \le u_2$. If $x_1 \le x_2$ and $x_1 \ne x_2$, then we will say that $x_1 < x_2$.

Remark 4.12. *1. It easy to see that under condition (**D1**), if $x_1 \le x_2$ then $\pi(t, x_1) \le \pi(t, x_2))$ for all $x_1, x_2 \in X_y$, $y \in Y$ and $t \in \mathbb{R}_+$.*

*2. If $(X, \mathbb{R}_+, \pi)$ is a skew-product dynamical system, i.e., $X = \mathbb{R}\times Y$ and $\pi = (\varphi, \sigma)$, then $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ satisfies condition (**D1**), if the cocycle φ is monotone increasing. This means that $x_1 \le x_2$ ($x_1, x_2 \in \mathbb{R}$) implies $\varphi(t, x_1, y) \le \varphi(t, x_2, y)$ for all $(t, y) \in \mathbb{R}_+ \times Y$.*

Let $y \in Y$. The fiber X_y is said to be distal, if

$$\inf_{t\in\mathbb{R}_+} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$$

for all $x_1, x_2 \in X_y$ such that $x_1 \ne x_2$.

Theorem 4.13. *Suppose that $M \subseteq X$ and the following conditions are fulfilled:*

- (i) *M is positively invariant and conditionally compact;*
- (ii) *the point $y \in Y$ is positively Poisson stable;*
- (iii) *$M_y := M \bigcap X_y \ne \emptyset$;*
- (iv) *the fiber M_y is distal.*

Then there exists a point $p_y \in M_y$ such that $\xi(p_y) = p_y$ for all $\xi \in \mathcal{E}_y^+$.

Proof. By Lemma 4.2 $\mathcal{E}_y^+$ is a nonempty, compact sub-semigroup of $M_y^{M_y}$. Under the conditions of Theorem 4.13 the set M_y is nonempty, compact and invariant with respect to semigroup $\mathcal{E}_y^+$. Now we will establish that every mapping $\xi \in \mathcal{E}_y^+$ is strictly increasing. In fact, let $x_1, x_2 \in M_y$ and $x_1 < x_2$.Since the fiber M_y is distal, then there exists a positive number $d = d(x_1, x_2)$ such that

$$pr_1(\pi(t, x_2) - \pi(t, x_1)) \ge d \qquad (10)$$

for all $t \in \mathbb{R}_+$, where by $pr_1(x)$ is denoted the first projection of $x \in X_y$. Let $\xi \in \mathcal{E}_y^+$ then there exists $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to +\infty$ as $n \to \infty$ and $\xi(x) = \lim_{n\to\infty} \pi(t_n, x)$

for all $x \in M_y$. From (10) we obtain $\xi(x_2) > \xi(x_1)$. Now to finish the proof of Theorem 4.13 it is sufficient to apply Theorem 3.12. □

Remark 4.14. *Note that under the conditions of Theorem 4.13 if the set M_y is $\mathcal{E}_y^+(M)$-minimal, then it consists of a single point.*

This statement can be proved using the same arguments as in the proof of Theorem 4.13 and taking into consideration Remark 3.13.

Let $y \in Y$. A point $x_0 \in X_y$ (respectively, a fiber X_y) is said to be uniformly stable, if for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon, x_0) > 0$ (respectively, there exists a positive number $\delta = \delta(\varepsilon) > 0$) such that $\rho(\pi(t_0, x), \pi(t_0, x_0)) < \delta$ (respectively, $\rho(\pi(t_0, x_1), \pi(t_0, x_2)) < \delta$ $(x_1, x_2 \in X_y)$) implies $\rho(\pi(t, x), \pi(t, x_0)) < \varepsilon$ (respectively, $\rho(\pi(t, x_1), \pi(t, x_2)) < \varepsilon$) for all $t \geq t_0$.

Lemma 4.15. *Suppose that the following conditions are fulfilled:*

(i) *the fiber X_y is a nonempty compact subset of X;*
(ii) *every point $x_0 \in X_y$ is uniformly stable.*

Then the fiber X_y is uniformly stable.

Proof. Let ε be an arbitrary positive number, $x_0 \in X_y$ and $\delta = \delta(\varepsilon, x_0)$ be a positive number chosen from the uniform stability of the point x_0. Then the family of open subsets $\{B(x_0, \delta)\}_{x_0 \in X_y}$ forms an open covering of X_y, where $B(x_0, \delta) := \{x \in X_y : \ \rho(x, x_0) < \delta\}$. Since X_y is a compact set and the metric space X is complete from the covering $\{B(x_0, \delta)\}_{x_0 \in X_y}$ we can extract a finite sub-covering $\{B(x_i, \delta)\}_{i=1}^k$. Denote by $\delta(\varepsilon) := \min\{\delta(\varepsilon, x_i) : \ i = 1, 2, \dots, k\}$, then it easy to check that $\rho(\pi(t_0, x_1), \pi(t_0, x_2)) < \delta$ $(x_1, x_2 \in X_y)$ implies $\rho(\pi(t, x_1), \pi(t, x_2)) < \varepsilon$ for all $t \geq t_0$. □

Theorem 4.16. *Suppose that $M \subseteq X$ and the following conditions are fulfilled:*

(i) *M is positively invariant and conditionally compact;*
(ii) *the point $y \in Y$ is positively Poisson stable;*
(iii) *$M_y := M \bigcap X_y \neq \emptyset$;*
(iv) *the fiber M_y is uniformly stable.*

Then there exists a point $p_y \in M_y$ such that $\xi(p_y) = p_y$ for all $\xi \in \mathcal{E}_y^+$.

Proof. Under the conditions of Theorem 4.16 the set M_y is nonempty, compact and invariant with respect to semigroup $\mathcal{E}_y^+$. By Lemma 3.8 there exists a $\mathcal{E}_y^+$-minimal subset $\mathcal{M} \subseteq M_y$. Denote by $\alpha := \min\{x \in \mathbb{R} |\ (x, y) \in \mathcal{M}\}$ and $\beta := \max\{x \in \mathbb{R} |\ (x, y) \in \mathcal{M}\}$. Since the set $\mathcal{M}$ is compact, then $(\alpha, y), (\beta, y) \in \mathcal{M}$. We will show that there exists a number $x \in K$ such that $p_y := (x, y) \in \mathcal{M}$ and $\xi(p_y) = p_y$ for all $\xi \in \mathcal{E}_y^+$. In fact. Since the set $\mathcal{M}$ is $\mathcal{E}_y^+$-minimal, then there exists $\xi \in \mathcal{E}_y^+$ such that $\xi(\alpha, y) = (\beta, y)$ or equivalently, $\xi_y(\alpha) = \beta$, where

$$\xi(x, y) := (\xi_y(x), y) \tag{11}$$

for all $x \in \mathcal{K}_y := \{x \in K : \text{such that } (x, y) \in \mathcal{M}\}$. Note that $\alpha \leq \xi_y(\alpha) \leq \xi_y(\beta) \leq \beta$ and the mapping $\xi_y : \mathcal{K}_y \mapsto \mathcal{K}_y$ is monotone increasing, i.e., $x_1 \leq x_2$ implies

$\xi_y(x_1) \le \xi_y(x_2)$ for all $x_1, x_2 \in \mathcal{K}_y$. Thus we have

$$\beta = \xi_y(\alpha) \le \xi_y(\beta) \le \beta$$

and, consequently, $\xi_y(\alpha) = \xi_y(\beta) = \beta$. On the other hand, if $x \in \mathcal{K}_y$, then $\beta = \xi_y(\alpha) \le \xi_y(x) \le \xi_y(\beta) = \beta$. Thus we obtain the equality

$$\xi_y(x) = \beta \tag{12}$$

for all $x \in \mathcal{K}_y$. Since $\xi \in \mathcal{E}_y^+$ (the mapping ξ is defined by equality (11)), then there exists a sequence $\{t_n\} \in \mathfrak{N}_y^{+\infty}$ such that

$$\pi(t_n, (x, y)) \mapsto (\beta, y) \tag{13}$$

as $n \to \infty$ for all $(x, y) \in \mathcal{M}$. From (13) and uniform stability of the fiber M_y it follows that

$$\lim_{t \to +\infty} \rho(\pi(t, (x, y)), \pi(t, (\beta, y))) = 0 \tag{14}$$

for all $x \in \mathbb{R}$ such that $(x, y) \in M_y$. From equality (14) and Theorem 4.3 we obtain that there exists a unique common fixed point $p_y \in \mathcal{M}$ of the semigroup $\mathcal{E}_y^+$. Taking into account (12) we conclude that $p_y = (\beta, y)$. Now we will show that $\alpha = \beta$. In fact, since the set $\mathcal{M}$ is $\mathcal{E}_y^+$-minimal, then there exists a mapping $\eta \in \mathcal{E}_y^+$ such that $\eta(\beta, y) = (\alpha, y)$ and, consequently, $\eta(\xi(\alpha, y)) = (\alpha, y)$. Since $\eta \circ \xi \in \mathcal{E}_y^+$ and under condition (14) p_y is a unique fixed point of the mapping $\eta \circ \xi \in \mathcal{E}_y^+$, then $(\alpha, y) = p_y$ and, consequently, $\alpha = \beta$, i.e., the minimal set $\mathcal{M}$ consists of a single point $\{p_y\}$. □

Remark 4.17. *For the non-autonomous dynamical systems $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ with two-sided base $(Y, \mathbb{T}, \sigma)$, i.e., in the case when $\mathbb{T} = \mathbb{R}$, we can prove that Theorem 4.16 follows from Theorem 4.13. In reality it can be proved that under the conditions of Theorem 4.16 from the uniform stability of the fiber M_y it follows its distality. In general case ($\mathbb{T} = \mathbb{R}_+$) this is an open question.*

Corollary 4.18. *Under the conditions of Theorem 4.16 there exists a point $p \in M_y$ which is comparable by character of recurrence with the point y.*

Proof. By Theorem 4.16 there exists a fixed point $p \in M_y$ of the semigroup $\mathcal{E}_y^+$. To prove this statement it is sufficient to show that the point p is required. In fact. Let $\{t_n\} \in \mathfrak{N}_y$, then $\{t_n\} \in \mathfrak{N}_p$. If we suppose that it is not true, then there are two subsequences $\{t_{n_k^i}\} \subset \{t_n\}$ $(i = 1, 2)$ such that $\lim_{k \to +\infty} \pi(t_{n_k^i}, p) = p_i$ (i=1,2) and $p_1 \ne p_2$. Without loss of generality we may suppose that the sequences $\{\pi(t_{n_k^i}, \cdot)\}$ are convergent in $M_y^{M_y}$. Denote by $\xi_i := \lim_{k \to +\infty} \pi(t_{n_k^i}, \cdot)$, then $\xi_i \in \mathcal{E}_y^+$ and we have $p_1 = \xi_1(p) = p = \xi_2(p) = p_2$. The obtained contradiction completes the proof our statement. □

Corollary 4.19. *Let $y \in Y$ be a stationary (respectively, τ-periodic, almost recurrent, Levitan almost periodic, recurrent, Poisson stable) point. Then under the conditions of Corollary 4.18 there exists a unique stationary (respectively, τ-periodic, almost automorphic, almost recurrent, Levitan almost periodic, recurrent, Poisson stable) point $p \in M_y$.*

Proof. This statement directly follows from Corollary 4.18 and Remark 4.5. □

Theorem 4.20. *Let M be a compact positively invariant subset of non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),\ (Y,\mathbb{T},\sigma),h\rangle$. Suppose that the following conditions are fulfilled:*

(i) *Y is a compact minimal set;*
(ii) *for every $q\in Y$ the fiber M_q is uniformly stable.*

Then there exists a point $p\in M_y$ which is uniformly comparable by the character of recurrence with $y\in Y$, i.e., $\mathfrak{M}_y\subseteq \mathfrak{M}_p$.

Proof. By Theorem 4.16 there exists a fixed point $p\in M_y$ of the semigroup $\mathcal{E}_y^+$. In view of Corollary 4.19 the point p is recurrent. To proof this statement it is sufficient to prove that the point p is required. In fact. Let $M := \overline{\{\pi(t,p)\ :t\in\mathbb{R}_+\}}$ then it is a compact minimal set because the point p is recurrent. We will show that $M_q := M\cap X_q$ (for all $q\in H(y) := \overline{\{\sigma(t,y)\ :t\in\mathbb{T}\}}$) consists of a single point. If we suppose that it is not true then there exist $q_0\in H(y)$ and $x_1,x_2\in M_{q_0}$ such that $x_1\neq x_2$. By Corollary 4.18 there exists a point $x_{q_0}\in M_{q_0}$ which is comparable by the character of recurrence with the point q_0. Without loss of generality we may suppose that $x_{q_0}=x_1$. Since the set M is minimal, then there exists a sequence $\{t_n\}\in\mathfrak{N}_{q_0}^{+\infty}$ such that $\{\pi(t_n,x_1)\}\to x_2$. On the other hand taking into consideration the inclusion $\mathfrak{N}_{q_0}^{+\infty}\subseteq\mathfrak{N}_{x_1}^{+\infty}$ we have $\{\pi(t_n,x_1)\}\to x_1$ and, consequently, $x_1=x_2$. The obtained contradiction prove our statement.

Now we will prove that $\mathfrak{M}_y\subseteq\mathfrak{M}_p$. Let $\{t_n\}\in\mathfrak{M}_y$, then $\{t_n\}\in\mathfrak{M}_p$. If we suppose that it is not true, then there are two subsequences $\{t_{n_k^i}\}$ $(i=1,2)$ such that $\lim\limits_{k\to+\infty}\pi(t_{n_k^i},p)=x_i$ (i=1,2) and $x_1\neq x_2$. Denote by $q_0 := \lim\limits_{n\to+\infty}\sigma(t_n,y)$, then $q_0\in H(y)$ and $x_1,x_2\in M_{q_0}$. On the other hand before it was proved that M_q consists of a single point for all $q\in H(y)$. The obtained contradiction completes the proof of Theorem. □

Remark 4.21. *Note that under the conditions of Theorem 4.20 if the set M_y is $\mathcal{E}_y^+$-minimal, then it consists of a single point $\{p\}$ which is uniformly comparable by character of recurrence with y.*

This statement can be proved using the same arguments as in the proof of Theorem 4.20 and taking into consideration Remark 3.13.

Corollary 4.22. *Let $y\in Y$ be a stationary (respectively, τ–periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent, Poisson stable) point. Then under the conditions of Theorem 4.20 there exists a stationary (respectively, τ–periodic, almost periodic, almost automorphic, recurrent, pseudo recurrent, Poisson stable) point $p\in M_y$ and $\mathfrak{M}_y\subseteq\mathfrak{M}_p$.*

Proof. This statement directly follows from Theorem 4.20 and Remark 4.9. □

Remark 4.23. *Note that for the almost periodic (respectively, almost automorphic) point y, the inclusion $\mathfrak{M}_y\subseteq\mathfrak{M}_p$ is equivalent to the inclusion $\mathcal{M}_p\subseteq\mathcal{M}_y$* [5], *where $\mathcal{M}_y$ is the Fourier modulus of the almost periodic (respectively, almost automorphic* [28, ChI]*) point y.*

A positively invariant subset $M \subseteq X$ is said to be distal with respect to non-autonomous dynamical systems $\langle (X,\mathbb{R}_+,\pi),\ (Y,\mathbb{T},\sigma),h\rangle$ if

$$\inf_{t\in\mathbb{R}_+} \rho(\pi(t,x_1),\pi(t,x_2)) > 0$$

for all $x_1,x_2 \in M_y$ $(x_1 \neq x_2)$ and $y \in Y$.

Theorem 4.24. *Let M be a compact positively invariant subset of non-autonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),\ (Y,\mathbb{T},\sigma),h\rangle$. Suppose that the following conditions are fulfilled:*

(i) *the point $y \in Y$ is recurrent;*
(ii) *the set M is distal.*

Then there exists a point $p \in M_y$ which is uniformly comparable by the character of recurrence with $y \in Y$.

Proof. By Theorem 4.13 there exists a fixed point $p \in M_y$ of the semigroup $\mathcal{E}_y^+$. Further this statement can be proved using absolutely the same arguments as in the proof of Theorem 4.20. □

Remark 4.25. *For the non-autonomous dynamical systems $\langle (X,\mathbb{T},\pi),\ (Y,\mathbb{R},\sigma),h\rangle$ with two-sided compact minimal base $(Y,\mathbb{R},\sigma)$ the following fact is well known (see, for example, B. M. Levitan and V. V. Zhikov* [17, ChVII]*): the compact positively invariant uniformly stable set $M \subseteq X$ is distal. From this fact it follows that Theorem 4.24 implies Theorem 4.20. In general case (i.e., when the base dynamical system $(Y,\mathbb{R}_+,\sigma)$ is one-sided) this question remains open.*

Remark 4.26. *1. Note that the algebraic approach using ideals and idempotents was originally proposed in the works of R. Ellis* [12].

2. Application of the Ellis semigroup theory to non-autonomous systems (non-autonomous ordinary differential equations, functional differential equations, partial differential equations) with compact base (driving system) has already been made in many works including those due to I. Bronsteyn [2]*, D. Cheban* [7]*, R. Ellis and R. Johnson* [13]*, R. Johnson* [16]*, R. Sacker and G. Sell* [19]-[21]*, G. Sell, W. Shen and Y. Yi* [24]*, W. Shen and Y. Yi* [28]*, V. Zhikov and B. Levitan* [17]*. As for the non-autonomous systems with noncompact base (driving system), the Ellis semigroup theory was applied in the works of D. Cheban* [6]-[8].

5. Scalar differential equations

In this section we study the problem of existence of Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable solutions for scalar differential equation of the form

$$x' = f(\sigma(t,y),x) \quad (y \in Y), \tag{15}$$

where $f \in C(Y \times \mathbb{R},\mathbb{R})$, Y is a complete metric space and $(Y,\mathbb{T},\sigma)$ is a dynamical system.

Recall [23] that the function $f \in C(Y \times \mathbb{R}, \mathbb{R})$ (respectively, equation (15)) is called regular, if for every $(x, y) \in \mathbb{R} \times Y$ equation (15) admits a unique solution $\varphi(t, x, y)$ defined on $\mathbb{R}_+$ passing through the point x at the initial moment, i.e., $\varphi(0, x, y) = x$.

A solution $\varphi(t, x_0, y)$ of equation (15) is called uniformly stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that $|\varphi(t_0, x, y) - \varphi(t_0, x_0, y)| < \delta$ $(t_0 \geq 0)$ implies $|\varphi(t, x, y) - \varphi(t, x_0, y)| < \varepsilon$ for all $t \geq t_0$.

Consider a cocycle $\langle \mathbb{R}, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ generated by equation (15) (see Example 2.8), the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ generated by cocycle φ (i.e., $X = \mathbb{R} \times Y$ and $\pi = (\varphi, \sigma)$) and the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ associated by φ, where $h = pr_2 : X \mapsto Y$.

Remark 5.1. *It easy to check that the solution $\varphi(t, x_0, y)$ of equation (15) is uniformly stable if and only if the point $(x_0, y) \in X_y$ of non-autonomous dynamical system (generated by cocycle φ) is uniformly stable.*

Lemma 5.2. *Suppose that $\varphi(t, x_0, y)$ is a bounded on $\mathbb{R}_+$ and uniformly stable solution of equation (15), then the following statements hold:*

- (i) *for all $q \in \omega_y$ every point $(x, q) \in X_q \bigcap \omega_{(x_0,y)}$ of non-autonomous dynamical system $\langle (X, \mathbb{T}, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ (generated by equation (15)) is uniformly stable, where $\omega_{(x_0,y)}$ is the ω-limit set of point (x_0, y);*
- (ii) *for every $q \in \omega_y$ the set $M_q := X_q \bigcap \omega_{(x_0,y)}$ is nonempty, compact and uniformly stable.*

Proof. Note that the first statement of Lemma was proved in [14, ChXI,Lemma 11.8] (see also [2, ChIV], [3] and [17, ChVII]) in the case, when the point y is almost periodic. In the general case it can be proved using the same arguments and we omit the details.

If $q \in \omega_y$, then it easy to see that under the conditions of Lemma the set M_q is nonempty. By Lemma 4.1 the set $H^+(x_0, y)$ (the closure of the positive semi-trajectory of the point (x_0, y)) is conditionally compact and, consequently, every set M_q is compact. According to the first item every point from M_q is uniformly stable. Since the set M_q is compact, then by Lemma 4.15 it is uniformly stable. □

Corollary 5.3. *Under the conditions of Lemma 5.2 if the point $y \in Y$ is positively Poisson stable, then the set $M_y := X_y \bigcap \omega_{(x_0,y)}$ is nonempty, compact and uniformly stable.*

Proof. This statement directly follows from Lemma 5.2 because in this case $y \in \omega_y$. □

Theorem 5.4. *Suppose that the following conditions are fulfilled:*

- (i) *the function $F \in C(Y \times \mathbb{R}, \mathbb{R})$ is regular;*
- (ii) *the point $y \in Y$ is positively Poisson stable;*
- (iii) *equation (15) admits a bounded on $\mathbb{R}_+$ solution $\varphi(t, x_0, y)$;*
- (iv) *the solution $\varphi(t, x_0, y)$ of equation (15) is uniformly stable.*

Then the following statements hold;

(i) *there exists at least one solution* $\varphi(t,p,y)$ *which is compatible by character of recurrence with the point* y*, i.e.,* $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_p^{+\infty}$*;*

(ii)

$$\lim_{t\to+\infty} |\varphi(t,x_0,y) - \varphi(t,p,y)| = 0 \tag{16}$$

Proof. Let $\langle \mathbb{R}, \varphi, (Y,\mathbb{T},\sigma)\rangle$ be a cocycle generated by differential equation (15) (see Example 2.8), $(X,\mathbb{R}_+,\pi)$ be the skew-product dynamical system generated by cocycle φ (i.e., $X = \mathbb{R}\times Y$ and $\pi = (\varphi,\sigma)$) and $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be the non-autonomous dynamical system associated by φ. Denote by $H^+(x_0,y)$ the closure of the positive semi-trajectory of the point $(x_0,y)\in X$. Note that the set $M := \omega_{(x_0,y)}$ is closed and positively invariant, by Lemma 5.2 under the conditions of Theorem 5.4 the set $M_y = X_y \bigcap \omega_{(x_0,y)}$ is nonempty, compact and uniformly stable. Now to finish the proof of the first statement it is sufficient to apply Theorem 4.16 and Corollary 4.18.

Now we will establish equality (16). Since $(p,y)\in\omega_{(x_0,y)}$, then there exists a sequence $\{t_n\}\in\mathfrak{N}_y^{+\infty}$ such that $\pi(t_n,(x_0,y))\to(p,y)$. Taking into consideration the inclusion $\mathfrak{N}_y^{+\infty}\subseteq\mathfrak{N}_{(p,y)}^{+\infty}$ we will have

$$|\varphi(t_n,x_0,y) - \varphi(t_n,p,y)| \to 0 \tag{17}$$

as $n\to\infty$. From (17) and the uniform stability of the set M_y we obtain (16). □

Recall (see, for example, [9, ChI]) that a function $\varphi\in C(\mathbb{R}_+,\mathbb{R})$ is said to be asymptotically stationary (respectively, τ-periodic, quasi-periodic, almost periodic, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable) if there exists a stationary (respectively, τ-periodic, quasi-periodic, almost periodic, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable) function $p\in C(\mathbb{R}_+,\mathbb{R})$ such that $\lim\limits_{t\to+\infty}|\varphi(t)-p(t)|=0$.

Corollary 5.5. *Under the conditions of Theorem 5.4 if the point* $y\in Y$ *is stationary (respectively,* τ*-periodic, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable), then the following statements hold:*

(i) *equation (15) admits at least one stationary (respectively,* τ*-periodic, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable) solution;*
(ii) *the solution* $\varphi(t,x_0,y)$ *is asymptotically stationary (respectively,* τ*-periodic, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable).*

Proof. This statement follows from Theorem 5.4, Corollary 4.19 and Theorem 2.11 (item 1.). □

Theorem 5.6. *Suppose that the following conditions are fulfilled:*

(i) *the function* $F\in C(Y\times\mathbb{R},\mathbb{R})$ *is regular;*
(ii) *the point* $y\in Y$ *is recurrent;*
(iii) *equation (15) admits a bounded on* $\mathbb{R}_+$ *solution* $\varphi(t,x_0,y)$*;*
(iv) *the solution* $\varphi(t,x_0,y)$ *of equation (15) is uniformly stable.*

Then the following statements hold:

(i) *there exists at least one solution $\varphi(t,p,y)$ which is uniformly compatible by character of recurrence with the point y, i.e., $\mathfrak{M}_y^{+\infty} \subseteq \mathfrak{M}_p^{+\infty}$ and $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_p^{+\infty}$;*

(ii)

$$\lim_{t\to+\infty} |\varphi(t,x_0,y) - \varphi(t,p,y)| = 0$$

Proof. This statement can be proved using the same arguments as in the proof of Theorem 5.4. But unless of Theorem 4.16 and Corollary 4.18 we need to apply in this case respectively Theorem 4.20 and Corollary 4.22. □

Corollary 5.7. *Under the conditions of Theorem 5.6 if the point $y \in Y$ is stationary (respectively, τ-periodic, quasi-periodic with the frequency $\nu = (\nu_1, \nu_2, \dots, \nu_m)$, almost automorphic, almost periodic, recurrent, pseudo recurrent), then the following statements hold:*

(i) *equation (15) admits at least one stationary (respectively, τ-periodic, quasi-periodic with the frequency $\nu = (\nu_1,\ \nu_2,\ \dots,\ \nu_m)$, almost automorphic, almost periodic, recurrent, pseudo recurrent) solution $\varphi(t,p,y)$. If the point $y \in Y$ is almost periodic (respectively, almost automorphic), then we have the inclusion of Fourier modulus $\mathcal{M}_p \subseteq \mathcal{M}_y$, where $\mathcal{M}_p$ is the Fourier modulus of the solution $\varphi(t,p,y)$.*

(ii) *the solution $\varphi(t,x_0,y)$ is asymptotically stationary (respectively, τ-periodic, quasi-periodic with the frequency $\nu = (\nu_1,\ \nu_2,\ \dots,\ \nu_m)$, almost automorphic, almost periodic, recurrent, pseudo recurrent).*

Proof. This statement follows from Theorem 5.6, Corollary 4.22 (see also Remark 4.23) and Theorem 2.11 (item 2.). □

Remark 5.8. *1. In the case, when $\mathbb{T} = \mathbb{R}$ and the point $y \in Y$ is almost periodic, Corollary 5.7 precises the results of V. V. Zhikov* [30] *(see also* [17, ChVII]*) and R. Sacker and G. Sell* [19] *(see also* [14, ChXI]*).*

2. In the case, when $\mathbb{T} = \mathbb{R}$ and the point $y \in Y$ is almost automorphic, Corollary 5.7 coincides with the result of W. Shen and Y. Yi [28, PartIII, Theorem B].

3. It is well known (see, for example, [14, ChXI] *and* [29, ChIII]*) that the bounded on $\mathbb{R}_+$ uniformly stable solution φ of periodic equation $x' = F(t,x)$ ($F \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ is a regular function) is asymptotically almost periodic. Thus Corollary 5.7 generalizes this statement for scalar almost periodic (respectively, almost automorphic, Levitan almost periodic, almost recurrent, recurrent, Poisson stable) equations. For two-dimensional almost periodic equations the last statement is false. The last fact can be confirmed by the following example: $z' = ia(t)z$, where $i^2 = -1$, $z \in \mathbb{C}$ and $a \in C(\mathbb{R},\mathbb{R})$ is an almost periodic function with unbounded primitive $A(t) := \int_0^t a(s)ds$. Every nontrivial solution of this equation is bounded (on $\mathbb{R}$) and uniformly stable, but this equation does not admit nontrivial asymptotically almost periodic (respectively, almost periodic* [2, ChIV]*) solutions.*

Let $\varphi(t, x_0, y)$ be a bounded on $\mathbb{R}_+$ and $Q := \overline{\varphi(\mathbb{R}_+, x_0, y)}$, where by bar is denoted the closure in $\mathbb{R}$. Let $q \in Y$. We will say that the bounded (on $\mathbb{R}_+$) solutions of equation

$$x' = F(\sigma(t, q), x) \tag{18}$$

are distal in Q, if for all $x_1, x_2 \in Q$ $(x_1 \neq x_2)$ with $\varphi(t, x_i, q) \in Q$ for all $t \geq 0$ we have $\inf\{|\varphi(t, x_1, q) - \varphi(t, x_2, q)| : \ t \in \mathbb{R}_+\} > 0$.

Remark 5.9. *1. Theorem 5.4 and Corollary 5.5 remain true if we replace the condition of uniform stability of solution $\varphi(t, x_0, y)$ by distality in Q of bounded solutions of equation (15).*

2. Theorem 5.6 and Corollary 5.7 remain true if we replace the condition of uniform stability of solution $\varphi(t, x_0, y)$ by distality in Q of bounded solutions of every equation (18) (for all $q \in Y$).

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