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I. U. BRONSHTEIN'S CONJECTURE FOR MONOTONE NONAUTONOMOUS DYNAMICAL SYSTEMS

DAVID CHEBAN

Abstract. In this paper we study the problem of almost periodicity of solutions for dissipative differential equations (Bronshtein's conjecture). We give a positive answer to this conjecture for monotone almost periodic systems of differential/difference equations.

1. Introduction

I. U. Bronshtein's conjecture [4, ChIV,p.273]**.** If an equation

$$x' = f(t,x) \quad (f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)) \tag{1}$$

with right hand side (Bohr) almost periodic in t satisfies the conditions of uniform positive stability and positive dissipativity, then it has at least one (Bohr) almost periodic solution.

Remark 1.1. *1. If $d \leq 3$, then the positive answer to this conjecture follows from the results of V. V. Zhikov* [34, ChII] *(see also* [17, ChVII] *and* [4, ChIV]*).*

2. Even for scalar equations ($d = 1$) as was shown by A. M. Fink and P. O. Frederickson [15] *(see also* [13, ChXII]*), dissipation (without uniform positive stability) does not imply the existence of almost periodic solutions.*

The aim of this paper is studying the problem of existence of Levitan/Bohr almost periodic (respectively, almost automorphic, recurrent and Poisson stable) solutions for dissipative differential equation (1), when the second right hand side is monotone with respect to spacial variable. The existence at least one quasi periodic (respectively, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent, Levitan almost periodic, almost recurrent, Poisson stable) solution of (1) is proved under the condition that every solution of equation (1) is positively uniformly Lyapunov stable.

The paper is organized as follows.

In Section 2 we collected some notions and facts from the theory of dynamical systems (the both autonomous and nonautonomous) which we use in this paper:

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Poisson stable motions and functions, cocycles, skew-product dynamical systems, monotone non-autonomous dynamical systems, Ellis semigroup.

Section 3 is dedicated to the study the global attractors of cocycles, when the phase space of driving system is noncompact.the structure of the ω-limit set of noncompact semitrajectory for autonomous and nonautonomous dynamical systems.

In Section 4 we formulate I. U. Bronshtein's conjecture for general nonautonomous dynamical systems. The positive answer for monotone nonautonomous dynamical systems is given (Theorem 4.2, Corollary 4.3 and Remark 4.4).

Section 5 is dedicated to the applications of our general results for differential (Theorems 5.15 and 5.16) and difference (Theorems 5.21 and 5.22) equations.

2. Some general properties of autonomous and nonautonomous dynamical systems

In this section we collect some notions and facts from the autonomous and non-autonomous dynamical systems [8] (see also, [10, Ch.IX]) which we will use below.

2.1. **Cocycles.** Let Y be a complete metric space, let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}_+ = \{t \in \mathbb{T} | \quad t \geq 0\}$ and $\mathbb{T}_- = \{t \in \mathbb{T} | \quad t \leq 0\}$. Let $(Y, \mathbb{T}, \sigma)$ be an autonomous two-sided dynamical system on Y and E be a real or complex Banach space with the norm $|\cdot|$.

Definition 2.1. *(Cocycle on the state space E with the base $(Y, \mathbb{T}, \sigma)$). The triplet $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$(or briefly ϕ) is said to be a cocycle (see, for example, [10] and [21]) on the state space E with the base $(Y, \mathbb{T}, \sigma)$ if the mapping $\phi : \mathbb{T}_+ \times Y \times E \to E$ satisfies the following conditions:*

(i) $\phi(0, y, u) = u$ *for all* $u \in E$ *and* $y \in Y$;
(ii) $\phi(t + \tau, y, u) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ *for all* $t, \tau \in \mathbb{T}_+, u \in E$ *and* $y \in Y$;
(iii) *the mapping* ϕ *is continuous.*

Definition 2.2. *(Skew-product dynamical system). Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on $E, X := E \times Y$ and π be a mapping from $\mathbb{T}_+ \times X$ to X defined by equality $\pi = (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, \omega, u), \sigma(t, y))$ for all $t \in \mathbb{T}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{T}_+, \pi)$ is an autonomous dynamical system and it is called [21] a skew-product dynamical system.*

Definition 2.3. *(Nonautonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two subsemigroup of the group $\mathbb{T}, (X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e., $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle (X, \mathbb{T}_1, \pi),\ (Y,\ \mathbb{T}_2,\ \sigma), h\rangle$ is called (see [4] and [10]) a nonautonomous dynamical system.*

Example 2.4. *(The nonautonomous dynamical system generated by cocycle ϕ.) Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{T}_+, \pi)$ be a skew-product dynamical system ($X = E \times Y, \pi = (\phi, \sigma)$) and $h = pr_2 : X \to Y$, then the triplet $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is a nonautonomous dynamical system.*

2.2. **Some general facts about nonautonomous dynamical systems.** In this subsection we give some general facts about nonautonomous dynamical systems without proofs. The more details and the proofs the readers can find in [8] (see also [10, Ch.IX]).

Definition 2.5. *Let (X, h, Y) be a fiber space, i.e., X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X into Y. The subset $M \subseteq X$ is said to be conditionally precompact* [8],[10, Ch.IX], *if the preimage $h^{-1}(Y') \bigcap M$ of every precompact subset $Y' \subseteq Y$ is a precompact subset of X. In particularly $M_y = h^{-1}(y) \bigcap M$ is a precompact subset of X_y for every $y \in Y$. The set M is called conditionally compact if it is closed and conditionally precompact.*

Definition 2.6. *Let $\langle E, \phi, (Y, \mathbb{T}, \sigma)\rangle$ (respectively, $(X, \mathbb{T}_+, \pi)$) be a cocycle (respectively, onesided dynamical system). The continuous mapping $\nu : \mathbb{T} \to E$ (respectively, $\gamma : \mathbb{T} \to X$) is called an entire trajectory of cocycle ϕ (respectively, of dynamical system $(X, \mathbb{T}_+, \pi)$) passing through the point $(u, y) \in E \times Y$ (respectively, $x \in X$) for $t = 0$ if $\phi(t, \nu(s), \sigma(s, y)) = \nu(t + s)$ and $\nu(0) = u$ (respectively, $\pi(t, \gamma(s)) = \gamma(t + s)$ and $\gamma(0) = x$) for all $t \in \mathbb{T}_+$and $s \in \mathbb{T}$.*

Denote by Φ_x the family of all entire trajectories of $(X, \mathbb{T}_+, \pi)$ passing through the point $x \in X$ at the initial moment $t = 0$ and $\Phi := \bigcup\{\Phi_x : \ x \in X\}$.

2.3. **Bohr/Levitan almost periodic, almost automorphic, recurrent and Poisson stable motions.**

Definition 2.7. *A number $\tau \in \mathbb{S}$ is called an $\varepsilon > 0$ shift of x (respectively, almost period of x), if $\rho(x\tau, x) < \varepsilon$ (respectively, $\rho(x(\tau + t), xt) < \varepsilon$ for all $t \in \mathbb{S}$).*

Definition 2.8. *A point $x \in X$ is called almost recurrent (respectively, Bohr almost periodic), if for any $\varepsilon > 0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x \in X$.*

Definition 2.9. *If the point $x \in X$ is almost recurrent and the set $H(x) := \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.*

Denote by $\mathfrak{N}_x := \{\{t_n\} : \ \{t_n\} \subset \mathbb{T}$ such that $\{\pi(t_n, x)\} \to x$ as $n \to \infty\}$.

Definition 2.10. *A point $x \in X$ of the dynamical system $(X, \mathbb{T}, \pi)$ is called Levitan almost periodic* [17], *if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.*

Definition 2.11. *A point $x \in X$ is called stable in the sense of Lagrange ($st.L$), if its trajectory $\Sigma_x := \Phi\{\pi(t, x) \ : \ t \in \mathbb{T}\}$ is relatively compact.*

Definition 2.12. *A point $x \in X$ is called almost automorphic in the dynamical system $(X, \mathbb{T}, \pi)$, if the following conditions hold:*

(i) *x is st.L;*
(ii) *the point $x \in X$ is Levitan almost periodic.*

Definition 2.13. *A point $x_0 \in X$ is called* [28, 29]

- *pseudo recurrent if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ and $p \in \Sigma_{x_0}$ there exist numbers $L = L(\varepsilon, t_0) > 0$ and $\tau = \tau(\varepsilon, t_0, p) \in [t_0, t_0 + L]$ such that $\tau \in \mathfrak{T}(p, \varepsilon)$);*

- *pseudo periodic (or uniformly Poisson stable) if for any $\varepsilon > 0$, $t_0 \in \mathbb{T}$ there exists a number $\tau = \tau(\varepsilon, t_0) > t_0$ such that $\tau \in \mathfrak{T}(p, \varepsilon))$ for any $p \in \Sigma_{x_0}$;*
- *Poisson stable in the positive (respectively, negative) direction if for any $\varepsilon > 0$ and $l > 0$ (respectively, $l < 0$) there exists a number $\tau > l$ (respectively, $\tau < l$) such that $\rho(\pi(\tau, x_0), x_0) < \varepsilon$. The point $x_0 \in X$ is called Poisson stable if it is stable (in the sense of Poisson) in the both directions.*

Remark 2.14. *1. Every pseudo periodic point is pseudo recurrent.*

2. If $x \in X$ is pseudo recurrent, then

- *it is Poisson stable;*
- *every point $p \in H(x)$ is pseudo recurrent;*
- *there exist pseudo recurrent points for which the set $H(x_0)$ is compact but not minimal* [25, ChV]*;*
- *there exist pseudo recurrent points which are not almost automorphic (respectively, pseudo periodic)* [25, ChV].

2.4. **B. A. Shcherbakov's principle of comparability of motions by their character of recurrence.** In this subsection we will present some notions and results stated and proved by B. A. Shcherbakov [25]-[28].

Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two dynamical systems.

Definition 2.15. *A point $x \in X$ is said to be comparable with $y \in Y$ by the character of recurrence, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of y is an ε–shift for x, i.e., $d(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$, where d (respectively, ρ) is the distance on Y (respectively, on X).*

Theorem 2.16. *Let x be comparable with $y \in Y$. If the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point $x \in X$ is so.*

Definition 2.17. *A point $x \in X$ is called uniformly comparable with $y \in Y$ by character of recurrence, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of $\sigma(t, y)$ is an ε–shift for $\pi(t, x)$ for all $t \in \mathbb{T}$, i.e., $d(\sigma(t+\tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t + \tau, x), x) < \varepsilon$ for all $t \in \mathbb{T}$ (or equivalently, $d(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{T}$).*

Denote by $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{R}\ :\ \text{such that } \{\pi(t_n, x)\} \text{ converges }\}$.

Definition 2.18. *A point $x \in X$ is said* [7],[9, ChII] *to be strongly comparable by character of recurrence with the point $y \in Y$, if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.*

Definition 2.19. *A point $y \in Y$ is said to be:*

(i) *stable in the sense of Lagrange in the positive direction (respectively, stable in the sense of Lagrange) if the set $H^+(y) := \overline{\{\sigma(t, y) |\ t \in \mathbb{T}_+\}}$ (respectively, $H(y) := \overline{\{\sigma(t, y) |\ t \in \mathbb{T}\}}$) is compact;*

(ii) *Poisson stable in the positive direction if $x \in \omega_x$, where*

$$\omega_x := \bigcap_{t\ge 0} \overline{\bigcup_{\tau \ge t} \pi(\tau, x)}\ .$$

Theorem 2.20. *Let X be a complete metric space. If the point x uniformly comparable by character of recurrence with y, then $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.*

Theorem 2.21. *Let y be stable in the sense of Lagrange. The inclusion $\mathfrak{M}_y \subseteq \mathfrak{M}_x$ takes place, if and only if the point x is stable in the sense of Lagrange and the point x uniformly comparable by character of recurrence with y.*

Theorem 2.22. *Let X and Y be two complete metric spaces, the point x be uniformly comparable with $y \in Y$ by the character of recurrence. If the point $y \in Y$ is recurrent (respectively, almost periodic, almost automorphic, uniformly Poisson stable), then so is the point $x \in X$.*

2.5. **Monotone Nonautonomous Dynamical Systems.** Let $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d :$ such that $x_i \ge 0$ $(x := (x_1, \dots, x_n))$ for any $i = 1, 2, \dots, d\}$ be the cone of nonnegative vectors of $\mathbb{R}^d$. By $\mathbb{R}^d_+$ on the space $\mathbb{R}^d$ is defined a partial order. Namely: $u \le v$ if $v - u \in \mathbb{R}^d_+$. Let $K \subset \mathbb{R}^d$ be a compact subset of $\mathbb{R}^d$, and for each $1 \le i \le d$, define $\alpha_i(K) := \min\{x_i |\ x = (x_1, \dots, x_d) \in K\}$ and $\beta_i(K) := \max\{x_i |\ x = (x_1, \dots, x_d) \in K\}$. Then $\alpha(K) := (\alpha_1(K), \dots, \alpha_d(K))$ and $\beta(K) := (\beta_1(K), \dots, \beta_d(K))$ are the greatest lower bound (*infimum*) and least upper bound (*supremum*) of with respect to the order on $\mathbb{R}^d$, respectively.

Definition 2.23. *Let $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle and $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a nonautonomous dynamical system associated by cocycle φ (i.e., $X := \mathbb{R}^d \times Y$, $\pi = (\varphi, \sigma)$ and $h := pr_2 : X \to Y$). The cocycle φ is said to be monotone if $u_1 \le u_2$ implies $\varphi(t, u_1, y) \le \varphi(t, u_2, y)$ for any $t > 0$ and $y \in Y$.*

Recall that a forward orbit $\{\pi(t, x_0)\ t \ge 0\}$ of non-autonomous dynamical systems $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be uniformly stable if for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\rho(\pi(t_0, x_0), \pi(t_0, x_0)) < \delta$ implies $d(\pi(t, x_0), \pi(t, x_0)) < \varepsilon$ for every $t \ge t_0$.

Below we will use the following assumptions:

- (C1) For every compact subset $K \subset X$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both the greatest lower bound (g.l.b.) $\alpha_y(K)$ and the least upper bound (l.u.b.) $\beta_y(K)$.
- (C2) For every $x \in X$, the semi-trajectory $\Sigma^+_x := \{\pi(t, x) :\ t \ge 0\}$ is conditionally precompact and its ω-limit set ω_x is positively uniformly stable.
- (C3) The non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ generated by cocycle φ is monotone.

Lemma 2.24. [11] *Assume that (C1)–(C3) hold, $x_0 \in X$ such that ω_{x_0} is positively uniformly stable. Let $K := \omega_{x_0}$ be fixed and $y_0 := h(x_0)$. Then if $q \in \omega_q \subseteq \omega_{y_0}$, $\alpha_q := \alpha_q(K)$, $K^1 := \omega_{\alpha_q}$, then the set $K^1_q := \omega_{\alpha_q} \bigcap X_q$ (respectively, $\omega_{\beta_q} \bigcap X_q$) consists a single point γ_q (respectively, δ_q), i.e., $K^1_q = \{\gamma_q\}$ (respectively, $\{\delta_q\}$).*

Theorem 2.25. [11] *Assume that (C1)–(C3) hold, $x_0 \in X$ such that ω_{x_0} is positively uniformly stable and $y_0 := h(x_0)$. Then the following statements hold:*

- (i) *if $y_0 \in \omega_{y_0}$, then the point γ_{y_0} (respectively, β_{y_0}) is comparable by character of recurrence with y_0 and*

(ii)
$$\lim_{n\to\infty}\rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0}))=0\ .$$

Corollary 2.26. *Under the conditions $(C1)-(C3)$ if the point y_0 is τ-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then:*

(i) *the point γ_{y_0} is so;*
(ii) *the point α_{y_0} is asymptotically τ-periodic (respectively, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically almost automorphic, asymptotically recurrent, asymptotically Poisson stable).*

Definition 2.27. *A point $x_0\in X$ is said to be:*

- *pseudo recurrent* [24, 28, 29] *if for any $\varepsilon>0$, $p\in\Sigma_{x_0}:=\{\pi(t,x_0):\ t\in\mathbb{T}\}$ and $t_0\in\mathbb{T}$ there exists $L=L(\varepsilon,t_0)>0$ such that*
$$B(p,\varepsilon)\bigcap\pi([t_0,t_0+L],p)\neq\emptyset,$$
where $B(p,\varepsilon):=\{x\in X:\ \rho(p,x)<\varepsilon\}$ and $\pi([t_0,t_0+L],p):=\{\pi(t,p):\ t\in[t_0,t_0+L]\}$;
- *uniformly Poisson stable* [2] *(or pseudo peiodic* [3, ChII,p.32]*) if for arbitrary $\varepsilon>0$ and $l>0$ there exists a number $\tau>l$ such that $\rho(\pi(t+\tau,x),\pi(t,x))<\varepsilon$ for any $t\in\mathbb{T}$.*

Remark 2.28. *1. Every recurrent (respectively, uniformly Poisson stable) point is pseudo recurrent. The inverse statement, generally speaking, is not true.*

2. If $x_0\in X$ is a pseudo recurrent point, then $p\in\omega_p$ for any $p\in H(x_0)$.

3. If x_0 is a Lagrange stable point and $p\in\omega_p$ for any $p\in H(x_0)$, then the point x_0 is pseudo recurrent.

Definition 2.29. *A point $x\in X$ is said to be strongly Poisson stable if $p\in\omega_p$ for any $p\in H(x)$.*

Remark 2.30. *Every pseudo recurrent point is strongly Poisson stable. The inverse statement, generally speaking, is not true.*

Theorem 2.31. [11] *Assume that $(C1)$–$(C3)$ hold, $x_0\in X$ and $y_0:=h(x_0)\in Y$ is strongly Poisson stable. Then the following statements hold:*

(i) *the point γ_{y_0} (respectively, δ_{y_0}) is strongly comparable by character of recurrence with y_0 and*
$$\lim_{t\to+\infty}\rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0}))=0.$$

Corollary 2.32. *Under the conditions $(C1)-(C3)$ if the point y_0 is τ-periodic (respectively, quasi periodic, Bohr almost periodic, recurrent, pseudo recurrent and Lagrange stable), then:*

(i) *the point u_{y_0} is so;*
(ii) *the point α_{y_0} is asymptotically τ-periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, pseudo recurrent).*

Remark 2.33. *1. If the point y_0 is recurrent (in the sense of Birkhoff), then Corollary 2.32 coincides with the results of the work of J. Jiang and X.-Q. Zhao* [**?**].

2. In the works of B. A. Shcherbakov [22]-[24], [25, ChV, Example 5.2.1] *were constructed examples of pseudo recurrent and Lagrange stable motions which are not recurrent (in the sense of Birkhoff).*

3. Global Attractors of Cocycles

Let W be a complete metric space.

Definition 3.1. *The family $\{I_y \mid y \in Y\}$ ($I_y \subset W$) of nonempty compact subsets W is called (see, for example,* [1] *and* [14]*) a compact pullback attractor (uniform pullback attractor) of a cocycle φ, if the following conditions hold:*

(i) *the set $I := \bigcup\{I_y \mid y \in Y\}$ is relatively compact;*
(ii) *the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ, i.e. $\varphi(t, I_y, y) = I_{\sigma(t,y)}$ for all $t \in \mathbb{T}_+$ and $y \in Y$;*
(iii) *for all $y \in Y$ (uniformly in $y \in Y$) and $K \in C(W)$*
$$\lim_{t\to+\infty} \beta(\varphi(t, K, y_{\sigma(-t,y)}), I_y) = 0,$$
where $\beta(A, B) := \sup\{\rho(a, B) : a \in A\}$ is a semi-distance of Hausdorff.

Definition 3.2. *The family $\{I_y \mid y \in Y\}(I_y \subset W)$ of nonempty compact subsets is called a compact (forward) global attractor of the cocycle φ, if the following conditions are fulfilled:*

(i) *the set $I := \bigcup\{I_y \mid y \in Y\}$ is relatively compact;*
(ii) *the family $\{I_y \mid y \in Y\}$ is invariant with respect to the cocycle φ;*
(iii) *the equality*
$$\lim_{t\to+\infty} \sup_{y\in Y} \beta(\varphi(t, K, y), I) = 0$$
holds for every $K \in C(W)$.

Let $M \subseteq W$ and
$$\omega_y(M) := \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, M, \sigma(-\tau, y))}$$
for any $y \in Y$.

Lemma 3.3. [10, ChII] *The following statements hold:*

(i) *The point $p \in \omega_y(M)$ if and only if there exit $t_n \to +\infty$ and $\{x_n\} \subseteq M$ such that $p = \lim_{n\to+\infty} \varphi(t_n, x_n, \sigma(-t_n, y))$;*
(ii) *$U(t, y)\omega_y(M) \subseteq \omega_{\sigma(t,y)}(M)$ for all $y \in Y$ and $t \in \mathbb{T}_+$, where $U(t, y) := \varphi(t, \cdot, y)$;*
(iii) *for any point $w \in \omega_y(M)$ the motion $\varphi(t, w, y)$ is defined on $\mathbb{S}$;*

(iv) *if there exits a nonempty compact $K \subset W$ such that*

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \sigma(-t, y)), K) = 0,$$

then $\omega_y(M) \neq \emptyset$, is compact,

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \sigma(-t, y)), \omega_y(M)) = 0$$

and

$$U(t, y)\omega_y(M) = \omega_{\sigma(t,y)}(M)$$

for all $y \in Y$ and $t \in \mathbb{T}_+$.

Definition 3.4. *A cocycle φ over $(Y, \mathbb{T}, \sigma)$ with the fiber W is said to be compactly dissipative, if there exits a nonempty compact $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(U(t, y)M, K) \mid y \in Y\} = 0 \tag{2}$$

for any $M \in C(W)$.

Theorem 3.5. [10, ChII] *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (2), then:*

1. *$I_y = \omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$ and*
$$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))K, I_y) = 0$$
for every $y \in Y$;
2. *$U(t, y)I_y = I_{yt}$ for all $y \in Y$ and $t \in \mathbb{T}_+$;*
3. $$\lim_{t\to+\infty} \beta(U(t, \sigma(-t, y))M, I_y) = 0$$
for all $M \in C(W)$ and $y \in Y$;
4. *the set I is relatively compact, where $I := \cup\{I_y \mid y \in Y\}$.*

Theorem 3.6. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative and K be the nonempty compact subset of W appearing in the equality (2), then the family of subsets $\{I_y | \ y \in Y\}$ is a maximal family possessing the properties 2.–4.*

Proof. Let $\{I_y^{'} | \ y \in Y\}$ be a family of subsets possessing properties 2.–4. Denote by $I^{'} := \bigcup\{I_y^{'} | \ y \in Y\}$ and $M := \overline{I^{'}}$, where by ar is denoted the closure of $I^{'}$. Since $M \in C(W)$, then for arbitrary $\varepsilon > 0$ and $y \in Y$ there exists a positive number $L(\varepsilon, y)$ such that

$$U(t, \sigma(-t, y))M \subseteq B(I_y, \varepsilon)$$

for any $t \geq L(\varepsilon, y)$. Note that $I_y^{'} = U(t, \sigma(-t, y))I_{\sigma(t,y)}^{'} \subseteq U(t, \sigma(-t, y))M \subseteq B(I_y, \varepsilon)$. Since ε is an arbitrary positive number we obtain $I_y^{'} \subseteq I_y$ for any $y \in Y$. □

Definition 3.7. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative, K be the nonempty compact subset of W appearing in the equality (2) and $I_y := \omega_y(K)$ for any $y \in Y$. The family of compact subsets $\{I_y | \ y \in Y\}$ is said to be a Levinson center (compact global attractor) of nonautonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$.*

Remark 3.8. *According to Theorem 3.6 by definition 3.7 is defined correctly the notion Levinson center (compact global attractor) for nonautonomous (cocycle) dynamical system $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$.*

Corollary 3.9. *Let $\langle W, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be compactly dissipative nonautonomous dynamical system, $\{I_y |\ y \in Y\}$ be its Levinson center and $\gamma : \mathbb{T} \mapsto W$ be a relatively compact full trajectory of φ (i.e., $\gamma(\mathbb{T})$ is relatively compact and there exists a point $y_0 \in Y$ such that $\gamma(t+s) = \varphi(t, \gamma(s), \sigma(s, y_0))$ for any $t \geq 0$ and $s \in \mathbb{T}$), then $\gamma(0) \in I_{y_0}$.*

Theorem 3.10. [10, ChII] *Under the conditions of Theorem 3.5 $w \in I_y$ $(y \in Y)$ if and only if there exits a whole trajectory $\nu : \mathbb{T} \to W$ of the cocycle φ, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{T})$ is relatively compact.*

Definition 3.11. *A family of subsets $\{I_y |\ y \in Y\}$ ($I_y \subseteq W$ for any $y \in Y$) is said to be upper semicontinuous if for any $y_0 \in Y$ and $y_n \to y_0$ as $n \to \infty$ we have*

$$\lim_{n\to\infty} \beta(I_{y_n}, I_{y_0}) = 0.$$

Lemma 3.12. *The following statements hold:*

- (i) *the family $\{I_y |\ y \in Y\}$ is invariant if and only if the set $J := \bigcup\{J_y |\ y \in Y\}$, where $J_y := I_y \times \{y\}$, is invariant with respect to skew-product dynamical system $(X, \mathbb{T}_+, \pi)$ ($X := W \times Y$ and $\pi := (\varphi, \sigma)$);*
- (ii) *if $\bigcup\{I_y |\ y \in Y\}$ is relatively compact, then the family $\{I_y |\ y \in Y\}$ is upper semicontinuous if and only if the set J is closed in X.*

Proof. The first statement is evident.

Suppose that the set $J \subseteq X$ is closed. If we suppose that the family $\{I_y |\ y \in Y\}$ is not upper semicontinuous, then there are $\varepsilon_0 > 0$, $y_0 \in Y$ and sequences $\{y_n\} \subset Y$ and $\{u_n\} \subset W$ such that $y_n \to y_0$ as $n \to \infty$, $u_n \in I_{y_n}$ and

$$\rho(u_n, I_{y_0}) \geq \varepsilon_0. \tag{3}$$

Since $\bigcup\{I_y |\ y \in Y\}$ is relatively compact, then without loss of generality we can suppose that the sequence $\{u_n\}$ is convergent. Denote by $u_0 := \lim\limits_{n\to\infty} u_n$ and passing into limit as $n \to \infty$ in inequality (3) we obtain $u_0 \notin I_{y_0}$. On the other hand we have $(u_n, y_n) \in J_{y_n} \subseteq J$ for any $n \in \mathbb{N}$ and since the set J is closed and $(u_n, y_n) \to (u_0, y_0)$ as $n \to \infty$, then $(u_0, y_0) \in J$ and, consequently, $u_o \in I_{y_0}$. The obtained contradiction proves our statement.

Let now the family $\{I_y\}$ upper semicontinuous and $(\bar{u}, \bar{y}) \in \overline{J}$. Then there exists a sequence $\{(u_n, y_n) \in J\}$ such that $(u_n, y_n) \to (\bar{u}, \bar{y})$. Since $u_n \in I_{y_n}$ and $\{I_y |\ y \in Y\}$ is upper semicontinuous, then $u_0 \in I_{y_0}$ and, consequently, $(u_0, y_0) \in J_{y_0} \subseteq J$. Thus the set J is closed. □

4. I. U. Bronshtein's conjecture for non-autonomous dynamical systems

Example 4.1. (*Bebutov's dynamical system*) Let X, W be two metric space. Denote by $C(\mathbb{T} \times W, X)$ the space of all continuous mappings $f : \mathbb{T} \times W \mapsto X$ equipped with the compact-open topology and σ be the mapping from $\mathbb{T} \times C(\mathbb{T} \times W, X)$ into $C(\mathbb{T} \times W, X)$ defined by the equality $\sigma(\tau, f) := f_\tau$ for all $\tau \in \mathbb{T}$ and $f \in C(\mathbb{T} \times W, X)$, where f_τ is the τ-translation (shift) of f with respect to variable t, i.e., $f_\tau(t, x) = f(t + \tau, x)$ for all $(t, x) \in \mathbb{T} \times W$. Then [10, Ch.I] the triplet $(C(\mathbb{T} \times W, X), \mathbb{T}, \sigma)$

is a dynamical system on $C(\mathbb{T}\times W, X)$ which is called a *shift dynamical system* (*dynamical system of translations* or *Bebutov's dynamical system*).

Recall that the function $\varphi \in C(\mathbb{T},\mathbb{R}^d)$ (respectively, $f \in C(\mathbb{T}\times\mathbb{R}^d,\mathbb{R}^n)$) possesses the property (A), if the motion $\sigma(\cdot,\varphi)$ (respectively, $\sigma(\cdot,f)$) generated by the function φ (respectively, f) possesses this property in the dynamical system $(C(\mathbb{T},\mathbb{R}^d),\mathbb{T},\sigma)$ (respectively, $(C(\mathbb{T}\times\mathbb{R}^d,\mathbb{R}^d),\mathbb{T},\sigma)$).

In the quality of the property (A) there can stand stability in the sense of Lagrange (st. L), uniform stability (un. st. $\mathcal{L}^+$) in the sense of Lyapunov, periodicity, almost periodicity, asymptotical almost periodicity and so on.

For example, a function $f \in C(\mathbb{T}\times\mathbb{R}^d,\mathbb{R}^d)$ is called almost periodic (respectively, recurrent etc) in $t\in\mathbb{R}$ uniformly with respect to (w.r.t.) w on every compact subset from $\mathbb{R}^d$, if the motion $\sigma(\cdot,f)$ is almost periodic (respectively, recurrent) in the dynamical system $(C(\mathbb{T}\times\mathbb{R}^d,\mathbb{R}^d),\mathbb{T},\sigma)$.

I. U. Bronshtein's conjecture for cocycles. Suppose that $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ is a cocycle under $(Y,\mathbb{T},\sigma)$ with the fiber W and the following conditions are fulfilled:

(i) the cocycle φ admits a compact global attractor $\mathbf{I} := \{I_y|\ y\in Y\}$;
(ii) the cocycle φ is positively uniformly Lyapunov stable, i.e., for any $\varepsilon>0$ and nonempty compact subset $K\subseteq W$ there exists a positive number $\delta=\delta(\varepsilon,K)$ such that $\rho(u_1,u_2)<\delta$ $(u_1,u_2\in K)$ implies $\rho(\varphi(t,u_1,y),\varphi(t,u_2,y))<\varepsilon$ for any $t\ge 0$ and $y\in Y$;
(iii) the dynamical system $(Y,\mathbb{T},\sigma)$ is minimal and Bohr almost periodic.

Then for any $y\in Y$ there exists at least one point $w_y\in I_y$ such that the motion $\varphi(t,u_y,y)$ is defined on entire axis $\mathbb{T}$ and it is Bohr almost periodic.

One of the main goal of this paper is a positive answer to I. U. Bronshtein's conjecture for monotone nonautonomous dynamical systems (Corollary 4.3).

Let $\langle\mathbb{R}^d,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle over $(Y,\mathbb{T},\sigma)$ with fiber $\mathbb{R}^d$.

Theorem 4.2. *Assume that the cocycle* φ

(i) *is monotone;*
(ii) *admits a compact global attractor* $\boldsymbol{I} := \{I_y|\ y\in Y\}$*;*
(iii) *is positively uniformly Lyapunov stable.*

Then the following statements hold:

(i) *if* $y_0\in\omega_{y_0}$*, then there exists a point* $a_{y_0}\in I_{y_0}$ *such that the full trajectory* γ_{y_0} *with* $x_0:=\gamma_{y_0}(0)=(a_{y_0},y_0)$ *is comparable by character of recurrence with* y_0*;*
(ii) *if* y_0 *is strongly Poisson stable, then the exists a point* $a_{y_0}\in I_{y_0}$ *such that the full trajectory* γ_{y_0} *with* $x_0:=\gamma_{y_0}(0)=(a_{y_0},y_0)$ *is strongly comparable by character of recurrence with* y_0.

Proof. Let $x_0=(u_{y_0},y_0)$, where u_{y_0} is an arbitrary point from W. Since the cocycle φ is compact dissipative, then the semitrajectory $\Sigma^+_{x_0} := \{(\varphi(t,u_{y_0},y_{y_0}),\sigma(t,y_0))|\ t\in$

$\mathbb{T}_+\}$ is conditionally precompact. It easy to check that under the conditions of Theorem the conditions $(C1)-(C3)$ are fulfilled. By Theorem 2.25 there exists at least one point $a_{y_0} \in W$ such that the full trajectory γ_{y_0} with $\gamma_{y_0}(0) = (a_{y_0}, y_0)$ is comparable by character of recurrence with y_0. According to Corollary 3.9 $a_{y_0} \in I_{y_0}$.

The second statement of Theorem can be proved using the same argument as in the proof of the first statement but instead of Theorem 2.25 we need to apply Theorem 2.31. □

Corollary 4.3. *Under the conditions of Theorem 4.2 the following statements take place:*

(i) *if the point y_0 is τ-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then there exist a point $a_{y_0} \in I_{y_0}$ such that the full trajectory $\gamma_{y_0} = (a_{y_0}, y_0)$ is so;*

(ii) *if the point y_0 is τ-periodic (respectively, quasi periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent and Lagrange stable), then there exist a point $a_{y_0} \in I_{y_0}$ such that the point $x_0 := (a_{y_0}, y_0)$ is so.*

Proof. This statement follows from the Theorems 4.2, 2.16 and 2.22. □

Remark 4.4. *Corollary 4.3 give as the positive answer for I. U. Bronshtein's conjecture for monotone Bohr almost periodic systems.*

5. Applications

5.1. **Dissipative Cocycles.** Let Y be a complete metric space (generally speaking noncompact), $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on the state space $\mathbb{R}^d$ and $(X, \mathbb{T}_+, \pi)$ be the corresponding skew-product dynamical system, where $X := \mathbb{R}^d \times Y$ and $\pi := (\varphi, \sigma)$.

Definition 5.1. *The cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is said to be dissipative if for any $y \in Y$ there is a positive number r_y such that*

$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < r_y$$

for any $y \in Y$ and $u \in \mathbb{R}^d$, i.e., for all $u \in \mathbb{R}^d$ and $y \in Y$ there exists a positive number $L(u, y)$ such that $|\varphi(t, u, y)| < r_y$ for any $t \geq L(u, y)$.

Definition 5.2. *The cocycle $\langle E, \varphi, (, \mathbb{T}, \sigma)\rangle$ is said to be uniformly dissipative if there exists a positive number r (r is not depend upon $y \in Y$) such that for any $R > 0$ there is a positive number $L(R)$ such that $|\varphi(t, u, y)| < r$ for all $y \in Y$ and $|u| \leq R$ and $t \geq L(R)$.*

Remark 5.3. *1. If the space E is finite-dimensional, then the uniformly dissipative is compactly dissipative.*

2. If the space E is finite-dimensional ($E = \mathbb{R}^d$) and Y is compact, then the dissipative system is uniformly dissipative [10, ChII].

Theorem 5.4. [10, ChIII] *Let Y be a compact and $\langle R^d, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle over the dynamical system $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^d$. Then the following statements are equivalent:*

1. *There exists a positive number R such that*
$$\limsup_{t\to+\infty} |\varphi(t, u, y)| < R$$
for all $u \in \mathbb{R}^d$ $y \in Y$.
2. *There is a positive number r_1 such that for all $u \in \mathbb{R}^d$ and $y \in Y$ there exists $\tau = \tau(u, y) > 0$ for which $|\varphi(\tau, u, y)| < r_1$.*
3. *There is a positive number r_2 such that*
$$\liminf_{t\to+\infty} |\varphi(t, u, y)| < r_2$$
for all $u \in \mathbb{R}^d$ and $y \in Y$.
4. *There exists a positive number R_0 and for all $R > 0$ there is $l(R) > 0$ such that $|\varphi(t, u, y)| \leq R_0$ for all $t \geq l(R)$, $u \in \mathbb{R}^d$, $|u| \leq R$ and $y \in Y$.*

Note that every condition 1.-4. that figures in Theorem 5.4 is equivalent to the dissipativity of the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ associated by the cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ over $(Y, \mathbb{T}, \sigma)$ with the fiber $\mathbb{R}^d$.

5.2. **Ordinary Differential Equations.** We will give below an example of a skew-product dynamical system which plays an important role in the study of non-autonomous differential equations.

Example 5.5. Consider the differential equation

$$u' = f(t, u), \tag{4}$$

where $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$. Along with the equation (4) we consider its H-class [4],[12], [17], [25],[28], i.e., the family of the equations

$$v' = g(t, v), \tag{5}$$

where $g \in H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}$ and $f_\tau(t, u) = f(t + \tau, u)$, where the bar indicating closure in the compact-open topology.

Condition (A1). The function $f \in C(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)$ is said to be regular if for every equation (5) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

We will suppose that the function f is regular. Denote by $\varphi(\cdot, v, g)$ the solution of (5) passing through the point $v \in \mathbb{R}^d$ for $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \times H(f) \to \mathbb{R}^d$ satisfies the following conditions (see, for example, [4],[19],[20]):

1) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^d$ and $g \in H(f)$;
2) $\varphi(t, \varphi(\tau, v, g), g_\tau) = \varphi(t + \tau, v, g)$ for each $v \in \mathbb{R}^d$, $g \in H(f)$ and $t, \tau \in \mathbb{R}_+$;
3) $\varphi : \mathbb{R}_+ \times E^n \times H(f) \to \mathbb{R}^d$ is continuous.

Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations on Y, induced by the dynamical system of translations $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{R}, \sigma)$. The triple $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is a cocycle over $(Y, \mathbb{R}_+, \sigma)$ with the fiber $\mathbb{R}^d$. Hence, the equation

(4) generates a cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ and the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi),\ (Y, \mathbb{R}, \sigma), h\rangle$, where $X := \mathbb{R}^d \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$.

Definition 5.6. *Recall that the equation (4) is called dissipative* [12], [18], [32], [33], *if for all $t_0 \in \mathbb{R}$ and $x_0 \in E^n$ there exists a unique solution $x(t; x_0, t_0)$ of the equation (4) passing through the point (x_0, t_0) and if there exists a number $R > 0$ such that $\lim\limits_{t\to+\infty} \sup |x(t; x_0, t_0)| < R$ for all $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$. In other words, for every solution $x(t; x_0, t_0)$ there is an instant $t_1 = t_0 + l(t_0, x_0)$, such that $|x(t; x_0, t_0)| < R$ for any $t \geq t_1$. If for any $r > 0$ the number $l(t_0, x_0)$ can be chosen independent on t_0 and x_0 with $|x_0| \leq r$, then the equation (4) is called uniformly dissipative* [12].

Below we will establish the relation between the dissipativity of the equation (4) and the dissipativity of the non-autonomous dynamical system generated by the equation (4).

Lemma 5.7. *Suppose that the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is regular. If equation (4) is uniformly dissipative, then the cocycle φ generated by equation (4) is also uniformly dissipative.*

Proof. Let (4) be uniformly dissipative, then there exists a positive number R such that for any $r > 0$ we can choose $l = l(r) > 0$ so that

$$|x(t; t_0, x_0)| < R \tag{6}$$

for any $t \geq t_0 + l(r)$. If $g \in H(f)$, then there exists a sequence $\{\tau_n\} \subset \mathbb{R}$ such that $f_{\tau_n} \to g$. Since $\varphi(t, x_0, f_\tau) = x(t + \tau; \tau, x_0)$ for any $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, then we have

$$|\varphi(t, x_0, f_{\tau_n})| = |x(t + \tau_n; \tau_n, x_0)| < R \tag{7}$$

for any $t \geq 0$, $|x_0| \leq r$ and $n \in \mathbb{N}$. Passing in limit in (7) as $n \to \infty$ we obtain $|\varphi(t, v, g)| \leq R$ for any $t \leq l(r)$, $|x_0| \leq r$ and $t \geq l(r)$. □

Lemma 5.8. [10, ChIII] *Let $f \in C(\mathbb{R} \times E^n, E^n)$ be regular. If $H(f)$ is compact, then equation (4) is uniformly dissipative if and if there is a positive number r such that*

$$\limsup_{t\to+\infty} |\varphi(t, x_0, g)| < r \quad (x_0 \in \mathbb{R}^d,\ g \in H(f)) .$$

Thus, for the equation (4) (f is regular and $H(f)$ is compact) we established that it is uniformly dissipative if and only if the non-autonomous dynamical system generated by this equation is dissipative.

Condition (A2). Equation (4) is monotone. This means that the cocycle $\langle \mathbb{R}^n, \varphi, (H(f),\ \mathbb{R},\ \sigma)\rangle$ (or shortly φ) generated by (4) is monotone, i.e., if $u, v \in \mathbb{R}^d$ and $u \leq v$ then $\varphi(t, u, g) \leq \varphi(t, v, g)$ for all $t \geq 0$ and $g \in H(f)$.

Let K be a closed cone in $\mathbb{R}^d$. The dual cone to K is the closed cone K^* in the dual space $\left(\mathbb{R}^d\right)^*$ of linear functions on $\mathbb{R}^d$, defined by

$$K^* := \{\lambda \in \left(\mathbb{R}^d\right)^* :\ \langle \lambda, x\rangle \geq 0 \text{ for any } x \in K\},$$

where $\langle \cdot, \cdot\rangle$ is the scalar product in $\mathbb{R}^d$.

Recall [30],[31, ChV] that the function $f \in C(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d)$ is said to be quasimonotone if for any $(t,u),(t,v) \in \mathbb{R}\times\mathbb{R}^d$ and $\phi \in \left(\mathbb{R}^d_+\right)^*$ we have: $u \le v$ and $\phi(u)=\phi(v)$ implies $\phi(f(t,u)) \le \phi(f(t,v))$.

Lemma 5.9. *Let $f \in C(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d)$ be a regular and quasimonotone function, then the following statements hold:*

(i) *if $u \le v$, then $\varphi(t,u,f) \le \varphi(t,v,f)$ for any $t \ge 0$;*
(ii) *any function $g \in H(f)$ is quasimonotone;*
(iii) *$u \le v$ implies $\varphi(t,u,g) \le \varphi(t,v,g)$ for any $t \ge 0$ and $g \in H(f)$;*
(iv) *equation (4) is monotone.*

Proof. The first statement is proved in [16, ChIII].

Let $g \in H(f)$, then there exists a sequence $\{h_k\} \subset \mathbb{R}$ such that $g(t,u) = \lim_{k\to\infty} f_{h_k}(t,u)$ for any $(t,u) \in \mathbb{R}\times\mathbb{R}^d$. Let $u \le v$ $(u,v \in \mathbb{R}^d)$ and $\phi \in \left(\mathbb{R}^n_+\right)^*$ such that $\phi(u)=\phi(v)$. Since f is quasimonote, then we will have

$$f_i(t+h_k,u) \le f_i(t+h_k,v) \tag{8}$$

and passing into limit in (8) as $k \to \infty$ we obtain that g is quasimonotone too.

Finally, the third statement follows from the first and second statements. Lemma is completely proved. □

Definition 5.10. *A solution $\varphi(t,u_0,f)$ of equation (4) is said to be:*

- *uniformly Lyapunov stable in the positive direction, if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\varphi(t_0,u,f) - \varphi(t_0,u_0,f)| < \delta$ $(t_0 \in \mathbb{R}$, $u \in \mathbb{R}^d)$ implies $|\varphi(t,x,f) - \varphi(t,x_0,f)| < \varepsilon$ for any $t \ge t_0$;*
- *compact on $\mathbb{R}_+$ if the set $Q := \overline{\varphi(\mathbb{R}_+,u_0,f)}$ is a compact subset of $\mathbb{R}^d$, where by bar is denoted the closure in $\mathbb{R}^d$ and $\varphi(\mathbb{R}_+,u_0,f) := \{\varphi(t,u_0,f) : t \in \mathbb{R}_+\}$.*

Let $f \in C(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d)$, $\sigma(t,f)$ be the motion (in the shift dynamical system $(C(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d),\mathbb{R},\sigma)$) generated by f, $u_0 \in \mathbb{R}^d$, $\varphi(t,u_0,f)$ be the solution of equation (4), $x_0 := (u_0,f) \in X := \mathbb{R}^d \times H(f)$ and $\pi(t,x_0) := (\varphi(t,u_0,f),\sigma(t,f))$ the motion of skew-product dynamical system $(X,\mathbb{R}_+,\pi)$.

Definition 5.11. *A solution $\varphi(t,u_0,f)$ of equation (4) is called* [9],[25],[28] *compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(t,x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t,f)$.*

Lemma 5.12. [6] *Let $\langle W,\varphi,(Y,\mathbb{T},\sigma)\rangle$ be a cocycle and $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ be a non-autonomous dynamical system associated by cocycle φ. Suppose that $x_0 := (u_0,y_0) \in X := W\times Y$ and the set $Q^+_{(u_0,y_0)} := \overline{\{\varphi(t,u_0,y_0) \mid t \in \mathbb{T}_+\}}$, where $\mathbb{T}_+ := \{t \in \mathbb{T} \mid t \ge 0\}$, is compact.*

Then the set $H^+(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{T}_+\}}$ is conditionally compact.

Remark 5.13. *If $x_0 := (u_0,y_0) \in X := W\times Y$ and α_{y_0} (respectively, γ_{y_0}) is a point from X defined in Lemma 2.24 then we denote by α_{u_0} (respectively, γ_{u_0}) a point from W such that $\alpha_{y_0} = (\alpha_{u_0},y_0)$ (respectively, $\gamma_{y_0} = (\gamma_{u_0},y_0)$).*

Definition 5.14. *A function f is said to be Poisson stable (respectively, strongly Poisson stable) in $t \in \mathbb{T}$ uniformly with respect to u on every compact subset of $\mathbb{R}^d$ if the point $f \in C(\mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d)$ is Poisson stable (respectively, strongly Poisson stable) in shift dynamical system $(C(\mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{T}, \sigma)$.*

Theorem 5.15. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^n d)$ is positively Poisson stable in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^d$;*
- *equation (4) is uniformly dissipative;*
- *each solution $\varphi(t, u_0, f)$ of equation (4) is positively uniformly Lyapunov stable.*

Then under conditions $(A1) - (A2)$ the following statement hold:

1. *equation (4) has at least one solution $\varphi(t, \gamma_{u_0}, f)$ defined and bounded on $\mathbb{R}$ which is compatible and belongs to Levinson center of (4).*
2. *if the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, Poisson stable) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then $\varphi(t, \gamma_{u_0}, f)$ is also stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, Poisson stable).*

Proof. Let $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ and $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^n d), \mathbb{R}, \sigma)$ be the shift dynamical system no $C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced by $(C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{R}, \sigma)$. Consider the cocycle $\langle \mathbb{R}^d, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ generated by equation (4) (see Condition (A1)). Now to finish the proof of Theorem it is sufficient to apply Theorems 4.2 (the first statement) and Corollary 4.3. Theorem is proved. □

Theorem 5.16. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is strongly Poisson stale in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^d$;*
- *equation (4) is uniformly dissipative;*
- *each solution $\varphi(t, u_0, g)$ of every equation (5) is positively uniformly Lyapunov stable.*

Then under conditions $(A1) - (A2)$ the following statements hold:

1. *every equation (5) has at least one solution $\varphi(t, \gamma_{v_0}, g)$ defined and bounded on $\mathbb{R}$ such that:*
    2. *solution $\varphi(t, \gamma_{v_0}, g)$ belongs to Levinson center of equation (5);*
    3. *$\varphi(t, \gamma_{v_0}, g)$ is a strongly compatible solution of (5).*
4. *if the function $f \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent and $H(f)$ is compact, uniformly Poisson stable and $H(f)$ is compact) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^d$, then $\varphi(t, \gamma_{u_0}, f)$ is also stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, uniformly Poisson stable).*

Proof. This statement can be proved similarly as Theorem 5.15 using Theorems 4.2 (the second statement) and Corollary 4.3. Theorem is proved. □

5.3. **Difference Equations.**

Example 5.17. Consider the equation

$$u_{n+1} = f(n, u_n), \tag{9}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Along with equation (9) we will consider H-class of (9), i.e., the family of equation

$$v_{n+1} = g(n, v_n), \ \ (g \in H(f)) \tag{10}$$

where $H(f) := \overline{\{f_\tau |\ \tau \in \mathbb{Z}\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$.

Denote by $\varphi(n, v, g)$ the solution of equation (10) with initial condition $\varphi(0, v, g) = v$. From the general properties of difference equations it follows that:

(i) $\varphi(0, v, g) = v$ for all $v \in \mathbb{R}^d$ and $g \in H(f)$;
(ii) $\varphi(n + m, v, g) = \varphi(n, \varphi(m, v, g), \sigma(m, g))$ for all $n, m \in \mathbb{Z}_+$ and $(v, g) \in \mathbb{R}^d \times H(f)$;
(iii) the mapping φ is continuous.

Thus every equation (9) generate a cocycle $\langle \mathbb{R}^d, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ over $(H(f), \mathbb{Z}, \sigma)$ with the fiber $\mathbb{R}^d$.

Lemma 5.18. *Let $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Suppose that the following conditions hold:*

(i) *$u_1, u_2 \in \mathbb{R}^d$ and $u_1 \le u_2$;*
(ii) *the function f is monotone non-decreasing with respect to variable $u \in \mathbb{R}^d$, i.e., $f(t, u_1) \le f(t, u_2)$ for any $t \in \mathbb{Z}$.*

Then $\varphi(n, v_1, g) \le \varphi(n, v_2, g)$ for any $n \in Z_+$, $v_1, v_2 \in \mathbb{R}^d$ with $v_1 \le v_2$ and $g \in H(f)$.

Proof. . Let $g \in H(f)$ and $v_1, v_2 \in \mathbb{R}^d$ with $v_1 \le v_2$, then there exists a sequence $\{\tau_k\} \subset \mathbb{Z}$ such that $g(t, v) = \lim\limits_{k\to\infty} f(t + \tau_k, v)$ uniformly w.r.t. v on every compact subset of $\mathbb{R}^d$. Since f is monotone in u, then we have

$$f(t + \tau_k, v_1) \le f(t + \tau_k, v_2) \tag{11}$$

for any $t \in \mathbb{Z}$ and $k \in \mathbb{N}$. Passing in limit as $k \to \infty$ in (11) we obtain $g(t, v_1) \le g(t, v_2)$ for any $t \in \mathbb{Z}$.

Let $v_1 \le v_2$ and $g \in H(f)$, then we have

$$\varphi(1, v_1, g) = g(0, v_1) \le g(0, v_2) = \varphi(1, v_2, g).$$

Suppose that $\varphi(s, v_1, g) \le \varphi(s, v_2, g)$ for all $ks \le t$, then we obtain

$$\varphi(s + 1, v_1, g) = g(1, \varphi(s, v_1, g)) \le g(1, \varphi(s, v_2, g)) = \varphi(s + 1, v_2, g).$$

□

Condition (D). Equation (9) is monotone. This means that the cocycle $\langle\mathbb{R}^d, \varphi, (H(f), \mathbb{Z}, \sigma)\rangle$ (or shortly φ) generated by (9) is monotone, i.e., if $u, v \in \mathbb{R}^n$ and $u \le v$ then $\varphi(t, u, g) \le \varphi(t, v, g)$ for all $t \ge 0$ and $g \in H(f)$.

Definition 5.19. *A solution $\varphi(t, u_0, f)$ of equation (9) is said to be:*

- *uniformly Lyapunov stable in the positive direction, if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\varphi(t_0, u, f) - \varphi(t_0, u_0, f)| < \delta$ ($t_0 \in \mathbb{Z}$, $u \in \mathbb{R}^d$) implies $|\varphi(t, x, f) - \varphi(t, x_0, f)| < \varepsilon$ for any $t \ge t_0$;*
- *compact on $\mathbb{Z}_+$ if the set $Q := \overline{\varphi(\mathbb{Z}_+, u_0, f)}$ is a compact subset of $\mathbb{R}^d$, where by bar is denoted the closure in $\mathbb{R}^n$ and $\varphi(\mathbb{Z}_+, u_0, f) := \{\varphi(t, u_0, f) : t \in \mathbb{Z}_+\}$.*

Let $f \in C(\mathbb{Z}\times\mathbb{R}^d, \mathbb{R}^d)$, $\sigma(t, f)$ be the motion (in the shift dynamical system $(C(\mathbb{Z}\times\mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$) generated by f, $u_0 \in \mathbb{R}^n$, $\varphi(t, u_0, f)$ be the solution of equation (9), $x_0 := (u_0, f) \in X := \mathbb{R}^d \times H(f)$ and $\pi(t, x_0) := (\varphi(t, u_0, f), \sigma(t, f))$ the motion of skew-product dynamical system $(X, \mathbb{Z}_+, \pi)$.

Definition 5.20. *A solution $\varphi(t, u_0, f)$ of equation (9) is called* [9],[25],[28] *compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(t, x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t, f)$.*

Theorem 5.21. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ is positively Poisson stable in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$;*
- *equation (9) is uniformly dissipative;*
- *each solution $\varphi(t, u_0, f)$ of equation (9) is positively uniformly Lyapunov stable.*

Then under condition (D) the following statement hold:

1. *equation (9) has at least one solution $\varphi(t, \gamma_{u_0}, f)$ defined and bounded on $\mathbb{Z}$ which is compatible and belongs to Levinson center of (9).*
2. *if the function $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, Poisson stable) in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^d$, then $\varphi(t, \gamma_{u_0}, f)$ is also stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, Poisson stable).*

Proof. Let $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ and $(C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$ be the shift dynamical system no $C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$. Denote by $Y := H(f)$ and $(Y, \mathbb{Z}, \sigma)$ the shift dynamical system on $H(f)$ induced by $(C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d), \mathbb{Z}, \sigma)$. Consider the cocycle $\langle\mathbb{R}^n, \varphi, (Y, \mathbb{Z}, \sigma)\rangle$ generated by equation (9). Now to finish the proof of Theorem it is sufficient to apply Theorems 4.2 (the first statement) and Corollary 4.3. Theorem is proved. □

Theorem 5.22. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{Z}\times\mathbb{R}^d, \mathbb{R}^d)$ is strongly Poisson stale in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^d$;*

- *equation (9) is uniformly dissipative;*
- *each solution $\varphi(t, u_0, g)$ of every equation (10) is positively uniformly Lyapunov stable.*

Then under condition (D) the following statements hold:

1. *every equation (10) has at least one solution $\varphi(t, \gamma_{v_0}, g)$ defined and bounded on $\mathbb{Z}$ such that:*
   2. *solution $\varphi(t, \gamma_{v_0}, g)$ belongs to Levinson center of equation (9);*
   3. *$\varphi(t, \gamma_{v_0}, g)$ is a strongly compatible solution of (10).*
4. *if the function $f \in C(\mathbb{Z} \times \mathbb{R}^d, \mathbb{R}^d)$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo recurrent and $H(f)$ is compact, uniformly Poisson stable and $H(f)$ is compact) in $t \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then $\varphi(t, \gamma_{u_0}, f)$ is also stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, uniformly Poisson stable).*

Proof. This statement can be proved similarly as Theorem 5.21 using Theorems 4.2 (the second statement) and Corollary 4.3. Theorem is proved. □

References

[1] Arnold L., *Random Dynamical Systems.* Springer-Verlag, 1998.

[2] V. M. Bebutov, On the shift dynamical systems on the space of continuous functions, *Bull. of Inst. of Math. of Moscow University* 2;5 (1940), pp.1-65. (in Russian)

[3] H. Bohr, *Almost Periodic Functions.* Chelsea Publishing Company, New York, 1947. ii+114 pp.

[4] Bronsteyn I. U., *Extensions of Minimal Transformation Group. Kishinev,* Stiintsa, 1974 (in Russian). [English translation: Extensions of Minimal Transformation Group, Sijthoff & Noordhoff, Alphen aan den Rijn. The Netherlands Germantown, Maryland USA, 1979]

[5] Caraballo Tomas and Cheban David, Almost Periodic Motions in Semi-Group Dynamic

[6] Caraballo T. and Cheban D., Almost Periodic and Almost Automorphic Solutions of Linear Differential/Difference Equations without Favard's Separation Condition.I *J. Differential Equations*, 246 (2009), pp.108–128.

[7] Cheban D. N., On the comparability of the points of dynamical systems with regard to the character of recurrence property in the limit. *Mathematical Sciences, issue N1,* Kishinev, "Shtiintsa", 1977, pp. 66-71.

[8] Cheban D. N., Global Pullback Atttactors of C-Analytic Nonautonomous Dynamical Systems. Stochastics and Dynamics. 2001,v.1 , No.4,pp.511-535.

[9] Cheban D. N., *Asymptotically Almost Periodic Solutions of Differential Equations.* Hindawi Publishing Corporation, New York, 2009, ix+186 pp.

[10] Cheban D.N. *Global Attractors of Nonautonomous Dynamical and Control Systems. 2nd Edition.* Interdisciplinary Mathematical Sciences, vol.18, River Edge, NJ: World Scientific, 2015, xxv+589 pp.

[11] David Cheban and Zhenxin Liu, Bohr/Levitan Almost Periodic, Almost Automorphic, Recurrent and Poisson Stable Solutions of Monotone Differential Equations. *Submitted, 2017.*

[12] Demidovich B. P., *Lectures on Mathematical Theory of Stability.* Moscow, "Nauka", 1967. (in Russian)

[13] A.M. Fink, *Almost Periodic Differential Equations,* Lecture notes in Mathematics, Vol. 377, Springer-Verlag, 1974.

[14] Flandoli F. and Schmalfuß B., Random Attractors for the Stochastic 3D Navier–Stokes Equation with Multiplicative White Noise, *Stochastics and Stochastics Reports*, **59** (1996), 21–45.

[15] Fink A. M. and Fredericson P. O., Ultimate Boundedness Does not Imply Almost Periodicity. *Journal of Differential Equations*, (9):280–284, 1971.

[16] M. Hirsch and H. Smith, *Monotone Dynamical Systems,* Handbook of Differential Equations, Vol. 2, Ch. IV. A. Canada, P. Drabek and A. Fonda, editors. Elsevier North Holland, 2005.

[17] Levitan B. M. and Zhikov V. V., *Almost Periodic Functions and Differential Equations.* Moscow State University Press, Moscow, 1978, 2004 p. (in Russian). [English translation: Almost Periodic Functions and Differential Equations. Cambridge Univ. Press, Cambridge, 1982, xi+211 p.]

[18] Pliss V. A. *Nonlocal Problems in the Theory of Oscilations.* Nauka, Moscow, 1964. (in Russian) [English translation: Nonlocal Problems in the Theory of Oscilations, Academic Press, 1966, 367 p.]

[19] Sell G. R., Non-Autonomous Differential Equations and Topological Dynamics, I. The basic theory. *Trans. Amer. Math. Soc.*, 127:241–262, 1967.

[20] Sell G. R., Non-Autonomous Differential Equations and Topological Dynamics, II. Limiting equations. *Trans. Amer. Math. Soc.*, 127:263–283, 1967.

[21] Sell G. R., *Lectures on Topological Dynamics and Differential Equations*, volume 2 of *Van Nostrand Reinhold math. studies*. Van Nostrand–Reinbold, London, 1971.

[22] Shcherbakov B. A., Classification of Poisson-stable motions. Pseudo-recurrent motions. *Dokl. Akad. Nauk SSSR,*146 (1962), pp.322==324.

[23] Shcherbakov B. A., On classes of Poisson stability of motion. Pseudorecurrent motions. *Bul. Akad. Stiince RSS Moldoven.,* 1963, no. 1, pp.58==72.

[24] Shcherbakov B. A., Constituent classes of Poisson-stable motions. *Sibirsk. Mat. Zh.,* Vol.5, No.6, 1964, pp.1397–1417. (in Russian) [English translation:].

[25] Shcherbakov B. A., *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations.* Ştiinţa, Chişinău, 1972, 231 p.(in Russian)

[26] Shcherbakov B. A., The compatible recurrence of the bounded solutions of first order differential equations. *Differencial'nye Uravnenija,* 10 (1974), 270–275.

[27] Shcherbakov B. A., The comparability of the motions of dynamical systems with regard to the nature of their recurrence. *Differential Equations* 11(1975), no. 7, 1246–1255.(in Russian) [English translation: Differential Equations, Vol.11. no.7, pp.937-943].

[28] B. A. Shcherbakov, *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations.* Ştiinţa, Chişinău, 1985, 147 pp. (in Russian)

[29] Sibirsky K. S., *Introduction to Topological Dynamics.* Kishinev, RIA AN MSSR, 1970, 144 p. (in Russian). [English translationn: Introduction to Topological Dynamics. Noordhoff, Leyden, 1975]

[30] Smith H. L., *Monotone Semiflows Generated by Functional Differential Equations. Journal of Differential Equations,* 66 (1987), pp.420-442.

[31] Smith H. L., *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems.* Series: Mathematical surveys and monographs, Volume 41. American Mathematical Society. Providence, Rhode Island, 1995, x+174 p.

[32] Yoshizawa T., Note on the Boundedness and the Ultimate Boundedness of Solutions $x' = f(t,x)$. *Memoirs of the College of Science. University of Kyoto, Serie A*, 29(3):275–291, 1955.

[33] Yoshizawa T., Lyapunov's Function and Boundedness of Solutions. *Funkcialaj Ekvacioj*, 2:95–142, 1959.

[34] Zhikov V. V., On Problem of Existence of Almost Periodic Solutions of Differential and Operator Equations. *Nauchnye Trudy VVPI, Matematika*, (8):94–188, 1969 (in Russian).

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